Orbits of linear groups

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Corresponding affine permutation group $H := VG \leq AGL(V)$, V = translation subgroup, $G = H_0$ stabilizer of zero vector. Orbits of G are *suborbits* of H, and H is primitive iff G is irreducible on V.

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For $\mathbb{F}$ finite this argument shows $\exists$ regular orbit if $\mathbb{F}$ finite and $|\mathbb{F}| > |G|$.

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Example 1 Let $G = S_c < GL_{c-1}(p) = GL(V)$, where $p > c$ and $V = \{(a_1, \ldots, a_c) : a_i \in \mathbb{F}_p, \sum a_i = 0\}$. Then G has regular orbits on V. Number of regular orbits is

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Example 2 Let $G = S_c \times C_2 < GL_{c-1}(p) = GL(V)$, where $p = c+1$, V as above, $C_2 = \langle -1_V \rangle$. Then G has no regular orbits on V.

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Eg. $G = S_c < GL_{c-1}(q)$, $\mathrm{char}(\mathbb{F}_q) > n$: here $f(q) = \frac{1}{c!}(q-1)(q-2)\cdots(q-c+1)$. This is a poly in q with roots equal to the exponents of the Weyl group $W(A_{c-1}) \cong S_c$.

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Same holds for all finite reflection groups in their natural representations (Orlik-Solomon). Eg. for $G = W(F_4) < GL_4(q)$, number of regular orbits

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Not always so nice, eg. for $G = PSL_2(7) < GL_3(q)$ (of index 2 in a unitary reflection group), $f(q) = \frac{1}{168}(q-1)(q^2+q-48)$.

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Eg. here's the table of marks for A_5:

	C_1	C_2	C_3	V_4	C_5	S_3	D_{10}	A_4	A_5
A_5/C_1	60								
A_5/C_2	30	2							
A_5/C_3	20		2						
A_5/V_4	15	3		3					
A_5/C_5	12				2				
A_5/S_3	10	2	1			1			
A_5/D_{10}	6	2			1		1		
A_5/A_4	5	1	2	1				1	
A_5/A_5	1	1	1	1	1	1	1	1	1

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Or $SL_2(5) \otimes (C_r.C_s) < GL_2(q) \otimes GL_r(q) < GL_{2r}(q)$
$(r|s-1,\ r, s > 5)$

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Conjecture *Let $G < GL_n(p) = GL(V)$, G a p'-group. The number of conjugacy classes $k(VG)$ in the semidirect product VG satisfies*

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"Simple type": G has an irreducible normal subgroup H such that $H/Z(H)$ is non-abelian simple.

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If $G < GL_n(p)$ is a p'-group of simple type, then G has a regular orbit unless one of:

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General classification of linear groups with/without regular orbits is out of reach at the moment. Need substitutes......

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(ii) $G \geq G_2(q)$ ($n = 6$, q even)
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Three, four,..... orbits: can be done if desperate

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(d) $G = S(q) < GL_2(q)$ (q odd), where

$$S(q) = \{ \begin{pmatrix} a & 0 \\ 0 & \pm a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & a \\ \pm a^{-1} & 0 \end{pmatrix} : a \in \mathbb{F}_q^* \}$$

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General case....???

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(b) G transitive, $p \,|\, |G| \Rightarrow G$ p-exceptional

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(a) G p-exceptional $\Rightarrow$ G has **no** regular orbit on V

(b) G transitive, $p \mid |G| \Rightarrow G$ p-exceptional

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$\exists$ many examples, eg. $V = W^k$, $G = H \,\mathrm{wr}\, K$ where $H \leq GL(W)$ is transitive on $W \backslash 0$ and $K \leq S_k$ has all orbits on the power set of $\{1, \ldots, k\}$ of p'-size. (Eg. K a p'-group)

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Yes, at least the irreducible ones ((Giudici, L, Praeger, Saxl, Tiep)......

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Theorem *Let $G \leq GL_d(p) = GL(V)$ be p-exceptional. Suppose G acts irreducibly and primitively on V. Then one of:*

(i) G transitive on $V \backslash 0$

(ii) $G \leq \Gamma L_1(p^n)$

(iii) $G = A_c, S_c < GL_{c-\epsilon}(2)$, $c = 2^r - 1$ or $2^r - 2$, $\epsilon = 1$ or 2

(iv) $G' = SL_2(5) < GL_4(3)$, *orbits* $1, 40, 40$
$PSL_2(11) < GL_5(3)$, *orbits* $1, 22, 110, 110$
$M_{11} < GL_5(3)$, *orbits* $1, 22, 220$
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Also have a classification of the imprimitive p-exceptional groups: $V = W^k$, $G \leq H \,\mathrm{wr}\, K$ where $H \leq GL(W)$ is transitive on $W\backslash 0$ and $K \leq S_k$ has all orbits on the power set of $\{1, \ldots, k\}$ of p'-size.

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Theorem *If $G \leq GL_d(p)$ is $\frac{1}{2}$-transitive and p divides $|G|$, then one of:*

(i) G is transitive on $V \backslash 0$

(ii) $G \leq \Gamma L_1(p^d)$

(iii) $G' = SL_2(5) < GL_4(3)$, orbits $1, 40, 40$.

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Gluck-Wolf theorem 1984 *Let p be a prime and G a finite p-soluble group. Suppose $N \lhd G$ and N has an irreducible character ϕ such that $\chi(1)/\phi(1)$ is coprime to p for all $\chi \subseteq \phi^G$. Then G/N has abelian Sylow p-subgroups.*

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Using our classification of p-exceptional groups, Tiep and Navarro have proved the Gluck-Wolf theorem for arbitrary finite groups G. May lead to the complete solution of the height zero conjecture.