Orbits of linear groups

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- Arithmetic conditions on orbit sizes, applications

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Corresponding affine permutation group $H := VG \le AGL(V)$, V = translation subgroup, $G = H_0$ stabilizer of zero vector. Orbits of G are *suborbits* of H, and H is primitive iff G is irreducible on V.

Let $G \leq GL_n(\mathbb{F}) = GL(V)$. Then G has a *regular* orbit on V if $\exists v \in V \setminus 0$ such that $G_v = 1$. Regular orbit is $v^G = \{vg : g \in G\}$, size |G|.

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For \mathbb{F} finite this argument shows \exists regular orbit if \mathbb{F} finite and $|\mathbb{F}| > |G|$.

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Far off. Delicate:

Example 1 Let $G = S_c < GL_{c-1}(p) = GL(V)$, where p > c and $V = \{(a_1, \ldots, a_c) : a_i \in \mathbb{F}_p, \sum a_i = 0\}$. Then G has regular orbits on V. Number of regular orbits is

$$\frac{1}{c!}(p-1)(p-2)\cdots(p-c+1).$$

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Example 2 Let $G = S_c \times C_2 < GL_{c-1}(p) = GL(V)$, where p = c + 1, V as above, $C_2 = \langle -1_V \rangle$. Then G has no regular orbits on V.

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Eg. $G = S_c < GL_{c-1}(q)$, $char(\mathbb{F}_q) > n$: here $f(q) = \frac{1}{c!}(q-1)(q-2)\cdots(q-c+1)$. This is a poly in q with roots equal to the exponents of the Weyl group $W(A_{c-1}) \cong S_c$.

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Not always so nice, eg. for $G = PSL_2(7) < GL_3(q)$ (of index 2 in a unitary reflection group), $f(q) = \frac{1}{168}(q-1)(q^2+q-48)$.

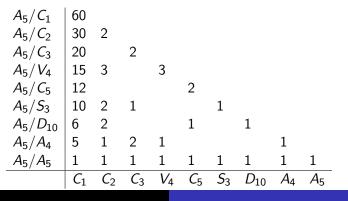
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Eg. here's the table of marks for A_5 :



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(r|s-1, r, s > 5)

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Conjecture Let $G < GL_n(p) = GL(V)$, G a p'-group. The number of conjugacy classes k(VG) in the semidirect product VG satisfies

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Robinson-Thompson reduction: conjecture proved if we can show that for G of "simple type" or "extraspecial type", there exists a regular orbit of G on V. "Simple type": G has an irreducible normal subgroup H such that

H/Z(H) is non-abelian simple.

Last one on regular orbits

Theorem (Hall-L-Seitz, Goodwin, Kohler-Pahlings, Riese) If $G < GL_n(p)$ is a p'-group of simple type, then G has a regular orbit unless one of:

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General classification of linear groups with/without regular orbits is out of reach at the moment. Need substitutes.....

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Hering's theorem Classification of transitive linear groups $G \leq GL_n(q)$: (i) $G \geq SL_n(q), Sp_n(q)$ (ii) $G \geq G_2(q)$ (n = 6, q even)(iii) $G \leq \Gamma L_1(q^n)$ (iv) ~ 10 exceptions, all with $|V| \leq 59^2$ (eg. $\mathbb{F}_{59}^* \circ SL_2(5) < GL_2(59)$).

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Three, four,..... orbits: can be done if desperate

Half-transitivity $G \leq GL_n(q) = GL(V)$ is $\frac{1}{2}$ -transitive if all orbits of G on $V \setminus 0$ have equal size. (Affine group $VG \leq AGL(V)$ is then $\frac{3}{2}$ -transitive.)

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General case....???

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Yes, at least the irreducible ones ((Giudici, L, Praeger, Saxl, Tiep).....

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Theorem Let $G \leq GL_d(p) = GL(V)$ be p-exceptional. Suppose G acts irreducibly and primitively on V. Then one of: (i) G transitive on V\0 (ii) $G \leq \Gamma L_1(p^n)$ (iii) $G = A_c, S_c < GL_{c-\epsilon}(2), c = 2^r - 1 \text{ or } 2^r - 2, \epsilon = 1 \text{ or } 2$ (iv) $G' = SL_2(5) < GL_4(3)$, orbits 1, 40, 40 $PSL_2(11) < GL_5(3)$, orbits 1, 22, 110, 110 $M_{11} < GL_5(3)$, orbits 1, 22, 220 $M_{23} < GL_{11}(2)$, orbits 1, 23, 253, 1771

($G \leq GL_n(p) = GL(V)$ is *p*-exceptional if *p* divides *G* and all orbits of *G* on *V* have size coprime to *p*.)

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Also have a classification of the imprimitive *p*-exceptional groups: $V = W^k$, $G \le H \text{ wr } K$ where $H \le GL(W)$ is transitive on $W \setminus 0$ and $K \le S_k$ has all orbits on the power set of $\{1, \ldots, k\}$ of *p*'-size.

Consequences

Recall G $\frac{1}{2}\text{-transitive} \Rightarrow$ all orbits on $V\backslash 0$ have same size \Rightarrow G is p-exceptional. Hence

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Theorem If $G \leq GL_d(p)$ is $\frac{1}{2}$ -transitive and p divides |G|, then one of:

(i) G is transitive on $V \setminus 0$ (ii) $G \le \Gamma L_1(p^d)$ (iii) $G' = SL_2(5) < GL_4(3)$, orbits 1,40,40.

Consequences

Gluck-Wolf theorem 1984 Let p be a prime and G a finite p-soluble group. Suppose $N \triangleleft G$ and N has an irreducible character ϕ such that $\chi(1)/\phi(1)$ is coprime to p for all $\chi \subseteq \phi^G$. Then G/N has abelian Sylow p-subgroups.

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This implies Brauer's *height zero* conjecture for *p*-soluble groups.

Using our classification of p-exceptional groups, Tiep and Navarro have proved the Gluck-Wolf theorem for arbitrary finite groups G. May lead to the complete solution of the height zero conjecture.