Permutation groups in MAGMA
Part II : the SubgroupLattice function

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Introduction

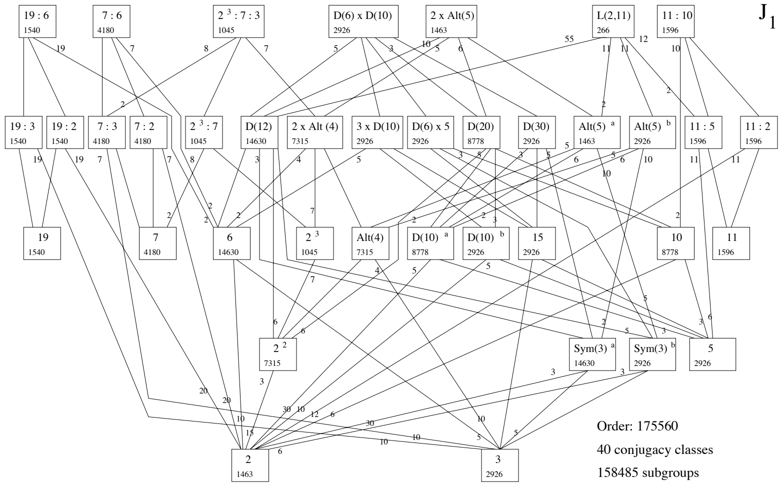
In this lecture, I intend to explain how the SubgroupLattice function of Magma works and show some possible improvements.

The subgroup lattice of a group

Two subgroups H and I of G are conjugate in G iff there exists an element g of G with $H^g := g * H * g^{-1} = I$

The subgroup lattice of a group G is a directed graph S whose vertices are representatives of the conjugacy classes of subgroups of G. Two vertices v and w are joined by a directed edge $[v, w]$ provided that there exist subgroups in the conjugacy class of subgroups represented by w which are maximal in a subgroup of the conjugacy class represented by v.

The subgroup lattice of J_1



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Subgroup lattices of sporadic groups

G	Factored order of G	Degree of G	Reference
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	11	Buekenhout, 1984
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	12	Buekenhout-Rees, 19
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	266	Buekenhout, 1984
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	22	Pfeiffer, 1997
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	100	Pahlings, 1987
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	23	Pfeiffer, 1997
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	100	
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	6156	Pfeiffer, 1991
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	24	Pfeiffer, 1997
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	275	Pfeiffer, 1997
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2058	Merkwitz, 1997
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	4060	
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	1782	
O′N	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	122760	Holt, 1998 (subgroup
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	276	Merkwitz, 1997
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	2300	
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	3510	
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	1140000	
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	8835156	
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	143127000	
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	31671	
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	98280	
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	173067389	
Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	920808	
BM	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	13571955000	
M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	97239461142009186000	

The SubgroupLattice function : historical limits

- Before 2005 : dependent on largest normal soluble subgroup of *G* and knowledge of perfect groups (Cannon, Cox, Holt).
Limit : J_2
- In 2002 : visit to Sydney and implementation based on MaximalSubgroups function that had been greatly improved by Holt and Cannon.
Available in Magma in 2005.
Limit : not really tested at that time but probably *Suz*
- In 2007 : wanted subgroup lattice of O'Nan : new improvement.

The SubgroupLattice function : what it tells you

Given your favorite group G, SubgroupLattice(G) will return you an object of type SubGrpLat.
This object consists of n objects that correspond to the n conjugacy classes of subgroups of G.
Lots of information is available like what are the overgroups of a group, what are the sizes of conjugacy classes, etc.

Basic algorithm

Start with a set classes containing just one element, namely the group G for which we want to compute the subgroup lattice.

While classes is nonempty, pick one element out of this set. Obviously, it is G the first time.

Compute its maximal subgroups and for each maximal M, add it to classes provided there is no subgroup in classes conjugate to M in G.

During that process, keep track of inclusions of respective subgroups considered.

At the end of this process, in classes there is one representative of each conjugacy class of subgroups of G. Moreover, we also have the maximal inclusions between classes. So the subgroup lattice is determined.

Problem 1 : how to implement your group ?

If you are lucky, the group you are interested in is available right away (using load or any other constructor).
Otherwise :
- presentations
- Atlas of Finite Groups
http://brauer.maths.qmul.ac.uk/Atlas/v3/

Problem 2 : get your hands on maximal subgroups

Again, if you are lucky, immediately available.
Otherwise :
- Atlas of Finite Groups (again)
http://brauer.maths.qmul.ac.uk/Atlas/v3/
- Sporadic conglomerator (Eamonn O'Brien)
http://www.math.auckland.ac.nz/ obrien/Sporadics/conglomerator.php

Improvement : reduce permutation degree

The RedPerm function, written by Bernd Souvignier
permits to reduce the permutation degree, speed up computations and reduce memory usage.
Example : in ONan, there are maximal subgroups isomorphic to L(3,7):2
We need to compute the maximal subgroups of these groups.
Degree 122760 : 15 seconds and more than 200Mb needed
Reduce their representation to degree say 5586
0.6 seconds and roughly 44 Mb needed, 37 of them to implement O'Nan, the stabilizer of a point, ...

Timings with this improvement

G	Order(G)	Deg(G)	$cc(G)$	$n(G)$	CPU Time
M_{11}	7,920	11	39	8,651	0.1s
M_{12}	95,040	12	147	214,871	0.41s
J_1	175,560	266	40	158,485	0.15s
M_{22}	443,520	22	156	941,627	0.47s
J_2	604,800	100	146	1,104,344	0.63s
M_{23}	10,200,960	23	204	17,318,406	0.8s
HS	44,352,000	100	589	149,985,646	5.09s
J_3	50,232,960	6156	137	71,564,248	17.46s
M_{24}	244,823,040	24	1529	1,363,957,253	73.94s
McL	898,128,000	275	373	1,719,739,392	4.51s
He	4,030,387,200	2058	1698	22,303,017,686	177.06s
Ru	145,926,144,000	4060	6035	963,226,363,401	**20117.720s**
Suz	448,345,497,600	1782	6381	4,057,939,316,149	**16130.870s**
O′N	460,815,505,920	122760	581	1,169,254,703,685	**7600s**
Co_3	495,766,656,000	276	2483	2,547,911,497,738	67.92s
Fi_{22}	64,561,751,654,400	3510	111004		**7.3 days**

A final parenthesis for Steve

Graphs