Permutation groups in MAGMA
Part I : the basics

SODO 2012
Dimitri Leemans
University of Auckland

Introduction

In Magma, we can work with several categories of finite (or infinite) groups, as, for instance

- Permutation groups : GrpPerm;
- Matrix groups (also infinite, cfr. Eamonn's lectures) : GrpMat;
- Polycyclic groups : GrpPC;
- Abelian groups (also infinite) : GrpAb;
- Finitely presented groups (cfr Marston's lectures): GrpFP;
- etc.

Construction of the Symmetric group $Sym(n)$

$\boxed{\text{SymmetricGroup(n)}}$

$\boxed{\text{Sym(n)}}$

Given an integer n, construct the group of all permutations of elements of the set $\{1, \dots, n\}$.

$\boxed{\text{SymmetricGroup(X)}}$

$\boxed{\text{Sym(X)}}$

Given a set X, construct the group of all permutations of the elements of X.

Construction of elements

$\boxed{\text{elt}< \text{G} \mid \text{L} >}$

Given a group G and a list L of objects defining an element g of G, contruct this element.

$\boxed{\text{G ! }(\ldots)(\ldots)\ldots(\ldots)}$

Given a group G and a permutation, construct the element corresponding to that permutation.

$\boxed{\text{Identity(G)}}$

$\boxed{\text{Id(G)}}$

$\boxed{\text{G ! 1}}$

Construct the identity element of G.

Arithmetic with permutations

$g * h$, $g\hat{}n$, $g/h = g * h^{-1}$, $g\hat{}h$, (g, h)
eq, ne, IsId(g), IsIdentity(g), Order(g), Degree(g), IsEven(g);

Construction of a permutation group

$\boxed{\text{PermutationGroup}< \text{X} \mid \text{L} >}$

$\boxed{\text{PermutationGroup}< \text{n} \mid \text{L} >}$

Suppose the set X has n elements. Construct the permutation group G acting on the set X generated by the permutations defined by the list L. An element of the list L must be of the following type :

1. a sequence of n elements of X defining a permutation of X;
2. a set or a sequence of sequences of type (1);
3. an element of $Sym(X)$;
4. a set or a sequence of elements of $Sym(X)$;
5. a subgroup of $Sym(X)$;
6. a set or sequence of subgroups of $Sym(X)$.

Construction of subgroups

$\boxed{\text{sub} < \text{G} \mid \text{L} >}$

Given the group G and L as above, construct the subgroup H of G generated by the elements specified by L.

Quotient groups

$\boxed{\text{G / N}}$

Given a group G and a normal subgroup N of G, construct the quotient group G/N.

Standard groups

$\boxed{\text{AbelianGroup(GrpPerm, Q)}}$

Construct the finite abelian group defined by the sequence $Q = [n_1, \ldots n_r]$ of natural numbers. The function returns the product of cyclic groups $C_{n_1} \times \ldots \times C_{n_r}$.

$\boxed{\text{AlternatingGroup(n)}}$

$\boxed{\text{Alt(n)}}$

Construct the alternating group of degree n.

$\boxed{\text{CyclicGroup(n)}}$

Construct the cyclic group of order n with generator $(1, \ldots, n)$.

$\boxed{\text{DihedralGroup(n)}}$

Construct the dihedral group of order $2n$ and degree n.

Construction of group extensions

$\boxed{\text{DirectProduct(G, H)}}$

Given two groups G and H, construct the direct product $G \times H$.
It is also possible to construct wreath products (type "WreathProduct;" in Magma for possible signatures).

Basic operations

$\boxed{\text{G.i}}$

Return the i^{th} generator of G.

$\boxed{\text{Degree(G)}}$

Return the degree of G.

$\boxed{\text{Generators(G)}}$

Return a set containing the generators of G.

$\boxed{\text{Generic(G)}}$

Construct the symmetric group in which G is naturally contained.

$\boxed{\text{Parent(g)}}$

The parent group of the permutation g.

$\boxed{\text{GSet(G)}}$

The natural set on which G acts as a permutation group.

Basic operations

$\boxed{\text{Order(G)}}$

$\boxed{\text{\#G}}$

The order of the group G.

$\boxed{\text{FactoredOrder(G)}}$

The factored order of G.

$\boxed{\text{Index(G,H)}}$

The index of H in G, provided H is a subgroup of G.

$\boxed{\text{FactoredIndex(G,H)}}$

The factored index.

$\boxed{\text{Representative(G)}}$

Return an element of G.

Some unary and binary operators

in, notin, subset, notsubset, eq, ne, . . .

Images, orbits, stabilizers

$\boxed{\texttt{x ^ g}}$
Given an element g of a group G with GSet X and an element $x \in X$, return the image of x under the action of g.

$\boxed{\texttt{x ^ G}}$
Given a permutation group (G, X), and $x \in X$, construct the orbit of x under the action of G.

$\boxed{\text{Orbits(G)}}$
Construct the orbits of G on its GSet.

$\boxed{\text{Stabilizer(G,y)}}$
Construct the stabilizer in G of y (this can be an element, a sequence, a set, etc.).

Conjugated elements

$\boxed{\text{IsConjugate(G,y,z)}}$

Test if y and z are conjugated elements of G.

$\boxed{\text{ConjugacyClasses(G)}}$

Give the conjugacy classes of elements of G.

Cosets, double cosets, etc.

$\boxed{\text{H * g}}$

The right coset Hg of the subgroup H of a group G (where $g \in G$).

$\boxed{\text{DoubleCoset(G,H,g,K)}}$

The double coset HgK of the subgroups H and K of the group G and $g \in G$.

$\boxed{\text{Transversal(G,H)}}$

Construct an indexed set T containing an element of each right coset of H in G. Also return the application $\phi : G \to T$ that associates to an element $g \in G$ the number of the right coset of H that contains it.

!!! For big transversals, we can use TransversalProcess. But BEWARE ! It works with LEFT cosets !

Standard construction of subgroups

$\boxed{\texttt{H ^ g}}$

Construt the conjugated subgroup H^g.

$\boxed{\texttt{H meet K}}$

Construct the subgroup $H \cap K$.

$\boxed{\texttt{Normalizer(G,H)}}$

Construct the normalizer of H in G.

$\boxed{\texttt{SylowSubgroup(G, p)}}$

Construct a p-Sylow subgroup of G. As they are all conjugate (by Sylow's theorem), it is easy to construct all of them using the Transversal function.

Subgroups

$\boxed{\text{SubgroupLattice}}$

Construct the lattice of subgroups of a group G.

Exemple : $Alt(5)$

This function is highly dependant of the MaximalSubgroups function. I will explain more later ...

$\boxed{\text{NormalSubgroups(G)}}$

Construct the normal subgroups of G.

Linear groups, and others

The following groups are immediately accessible :
AGL, ASL, AGammaL, ASigmaL
PGL, PSL, PGammaL, PSigmaL
PGU, PSU, PGammaU, PSigmaU
PSp, PSigmaSp
PGO, PGOPlus, PGOMinus, PSO, PSOPlus, PSOMinus
POmega, POmegaPlus, POmegaMinus
PSz
Some sporadic groups are also immediately accessible using the "load" command : Co_2, Co_3, He, HS, J_1, J_2, J_3, M_i $(i = 11, 12, 22, 23, 24)$, McL, Ru, Suz.
For those not accessible, look at the atlas of finite groups (or other sources), and use, for instance finitely presented groups ...

Parenthesis on finitely presented groups

When a group is not directly accessible in MAGMA and you don't have generating permutations for that group, but you know a presentation for that group, you can construct it as a finitely presented group and then convert it into a permutation group (as long as this group is finite).
Example : the O'Nan sporadic simple group of order 460.815.505.920, whose smallest permutation representation is on 122.760 points.
Computing time on this laptop (MacBook Air, 2.13Ghz Intel Core Duo, 2Gb DDR3) : less than 3 seconds.

Databases of groups

All transitive groups of degree $\leq$ TransitiveGroupDatabaseLimit().
Acces : TransitiveGroup(d,n)
All primitive groups of degree $\leq$ PrimitiveGroupDatabaseLimit().
Acces : PrimitiveGroup(d,n)
All groups of order $\leq$ SmallGroupDatabaseLimit().
Acces : SmallGroup(n,m), SmallGroups(n)