Reflexibility of regular Cayley maps on dihedral groups

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February 13, 2012, Queenstown

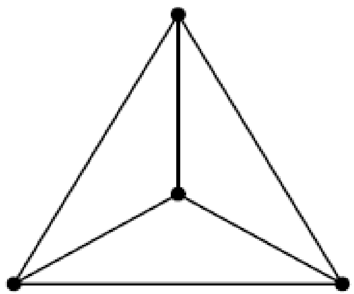
Outline

1. Introduction to maps and regular maps
2. Introduction to Cayley graphs, Cayley maps and regular Cayley maps
3. Reflexible Cayley maps on dihedral groups
4. Some remarks

Introductions to maps and regular maps

[Definition]

1. A (topological) map $\mathfrak{M}=G \rightarrow S$ is a 2-cell embedding of a graph G into a closed surface S.



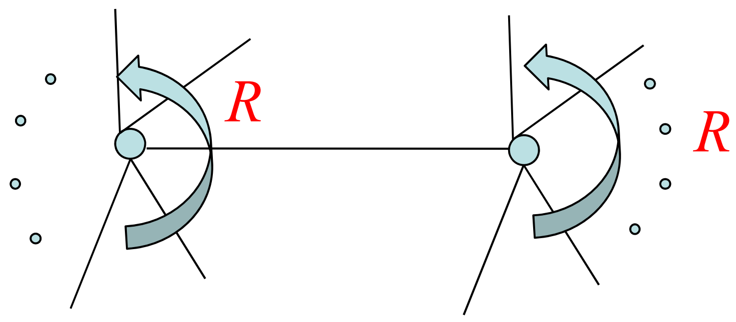
2. For any map $\mathfrak{M}=G \rightarrow S$, a mutually incident vertex-edge pair is called an *arc* of $\mathfrak{M}$. The set of arcs of $\mathfrak{M}$ is denoted by D($\mathfrak{M}$).

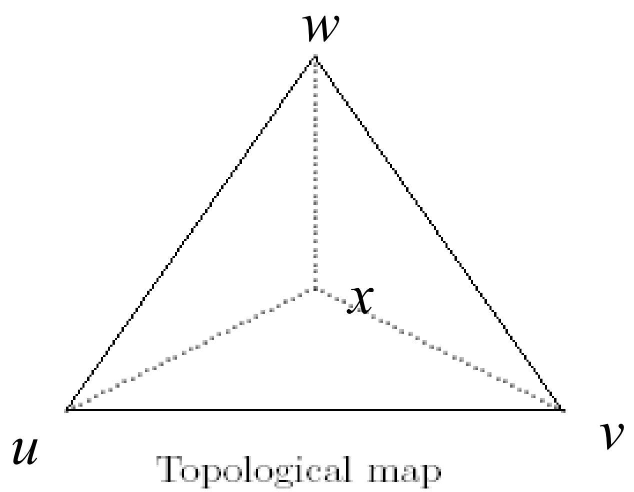
3. Any orientable map $\mathfrak{M} = G \to S$ can be descibed by a pair (G,R) such that

(1) G is an undirected graph.

(2) R is a permutation of $D(G)$ whose orbits coincide with the sets of arcs based at the same vertex.

The permutation R is called rotation of $\mathfrak{M}$.





$$R = (uv\ uw\ ux)(vu\ vx\ vw)(wu\ wv\ wx)(xu\ xw\ xv)$$

4. For two maps $\mathfrak{M}_1 = G_1 \to S_1$ and $\mathfrak{M}_2 = G_2 \to S_2$, a *map isomorphism* from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ is a graph isomorphism from G_1 to G_2 which can be extended to a surface homeomorphism from S_1 to S_2 in the embeddings.

If $G_1 = G_2 = G$, a map isomorphism is called a *map automorphism*. The set of automorphisms of $\mathfrak{M}$ is denoted by Aut($\mathfrak{M}$).

5. For any map $\mathfrak{M} = G \to S$ with an orientable surface S, the set of orientation-preserving(orientation-reversing, resp.) automorphism is denoted by $\text{Aut}^{+}(\mathfrak{M})$ ($\text{Aut}^{-}(\mathfrak{M})$, resp.).

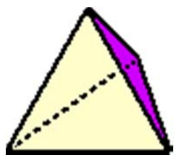
[Some properties]

1. If the valency of $\mathfrak{M}$ is greater than two then $\text{Aut}(\mathfrak{M})$ is a faithful subgroup of Aut(G).

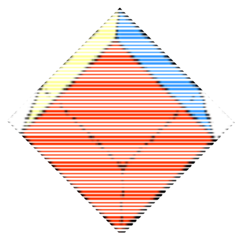
2. For any map $\mathfrak{M} = G \to S$ with an orientable surface S, $\text{Aut}^{+}(\mathfrak{M})$ acts semi-regularly on D(G), the set of arcs of G. If the action is regular then we call $\mathfrak{M}$ an *regular map* or an *regular embedding* of G.

3. If $\mathfrak{M}$ and $\mathfrak{M}^{-1} = (G : R^{-1})$ are isomorphic then $\mathfrak{M}$ is called reflexible. Otherwise, $\mathfrak{M}$ is called chiral.
 Note that $\mathfrak{M} = G \to S$ is reflexible iff $\text{Aut}^{-}(\mathfrak{M}) \neq \varnothing$.

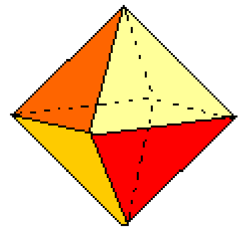
Five Platonic Solids



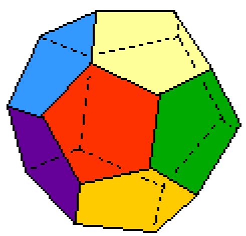
Tetrahedron



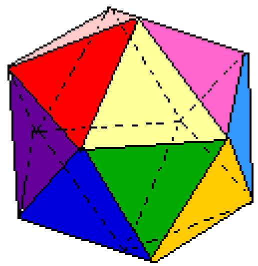
Hexahedron



Octahedron



Dodecahedron



Icosahedron

Cayley graphs, Cayley maps and regular Cayley mapss

[Definition]

1. For a group Γ and a set $X \subset \Gamma$ such that $X^{-1} = X$,

a Cayley graph $\mathrm{Cay}(\Gamma : X) = (V, E)$ is a graph such that

(1) $V = \Gamma$ and

(2) $E = \{\{g, gx\} \mid x \in X\}$.

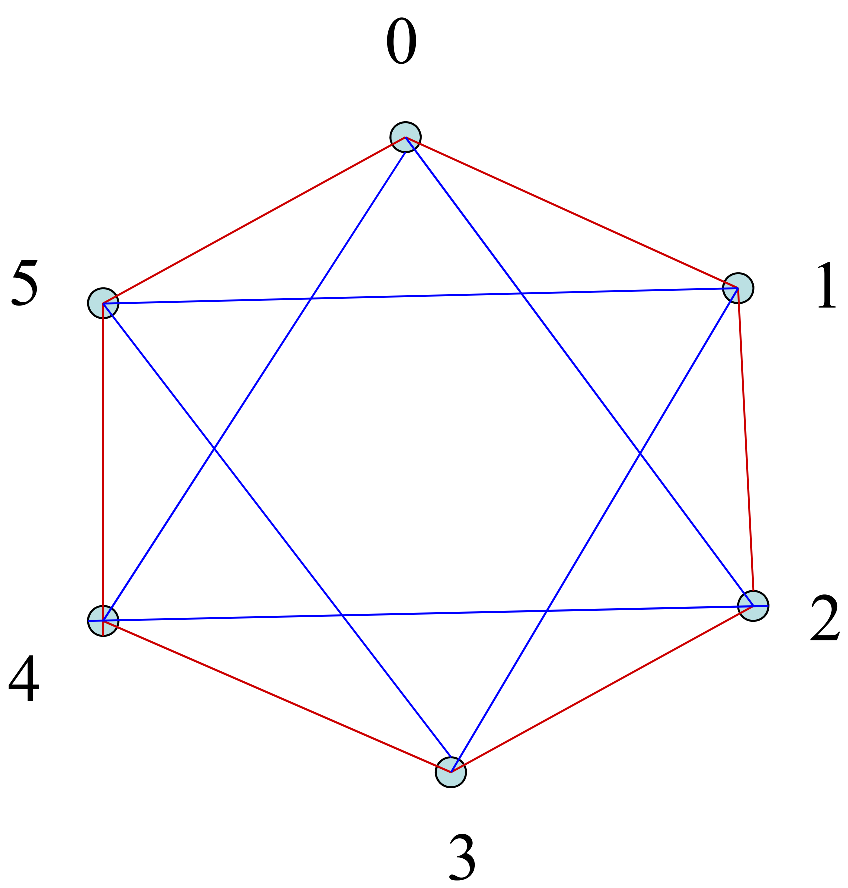
2. For any $g \in \Gamma$, let $L_g : \Gamma \to \Gamma$ such that

$L_g(h) = gh$ for any $h \in \Gamma$. Let $L_\Gamma = \{L_g \mid g \in \Gamma\}$.

Note that $L_\Gamma \leq \mathrm{Aut}(Cay(\Gamma : X))$ for any Cayley graph $Cay(\Gamma : X)$.

Example:

$$G = Cay(\mathbb{Z}_6 : \{1,2,4,5\})$$



1 , 5

2 , 4

3. For a Cayley graph $G = Cay(\Gamma : X)$ and cyclic permutation p of X, a Cayley map $\mathrm{CM}(\Gamma : X, p)$ is a map $\mathfrak{M} = (G, R)$ such that

$R(g, gx) = (g, gp(x))$ for any $g \in \Gamma$ and $x \in \Gamma$.

Note that $L_\Gamma \le \mathrm{Aut}^+(CM(\Gamma : X, p))$ for any Cayley map $CM(\Gamma : X, p)$.

4. For a Cayley map $CM(\Gamma : X, p)$ and for any $x \in X$,

if $p(x)^{-1} = p(x^{-1})$ then $CM(\Gamma : X, p)$ is called balanced

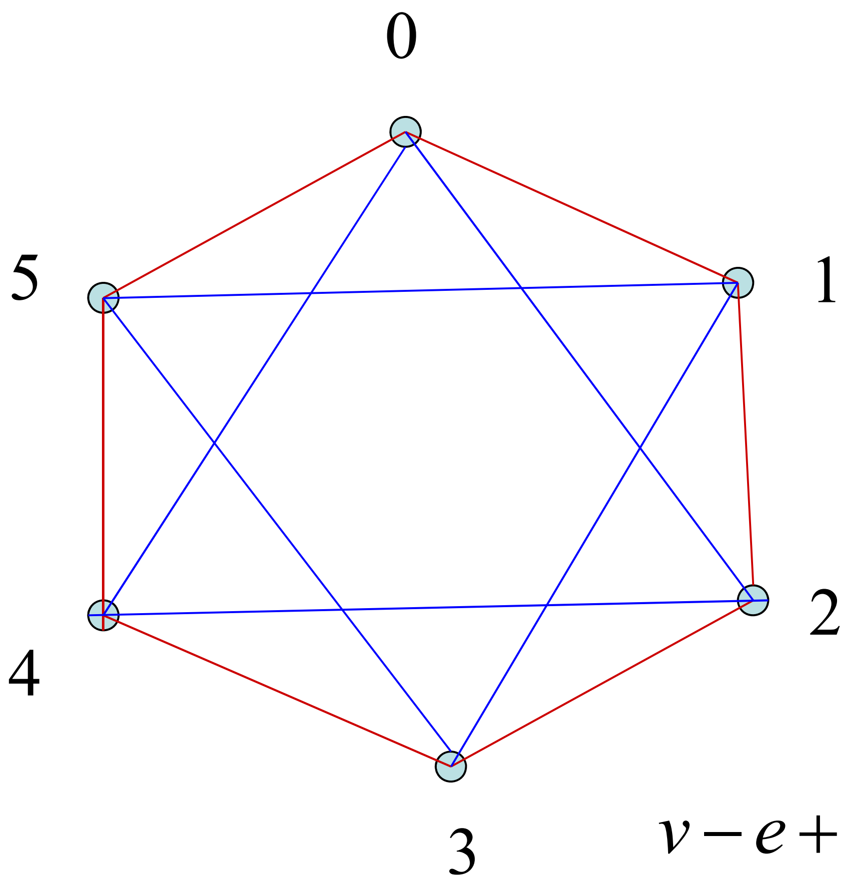
if $p(x)^{-1} = p^{-1}(x^{-1})$ then anti-balanced

if $p(x)^{-1} = p^{t}(x^{-1})$ then t-balanced.

$$p(x)^{-1} = p^{t}(x^{-1}) \Rightarrow p^{t}(x)^{-1} = p^{t^2}(x^{-1}) \Rightarrow t^2 \equiv 1 \ (\mathrm{mod}\ |X|)$$

Example:

$G = Cay(\mathbb{Z}_6 : \{1,2,4,5\}, p=(1,2,4,5))$



$triangle : 2$

$hexagon : 1$

$12-gon : 1$

$v - e + f = 6 - 12 + 4 = -2$

supporting surface: double torus

Automorphisms of Cayley maps

For a Cayley map $CM(\Gamma : X, p)$ with $p = (x_0, x_1, \ldots, x_{d-1})$, let $c : \{0,1,\ldots,d-1\} \to \{0,1,\ldots,d-1\}$ be a bijection such that $x_i^{-1} = x_{c(i)}$ for any $i \in \{0,1,\ldots,d-1\}$.

[Example]

$$Cay(\mathbb{Z}_6 : \{1,2,4,5\}, p = (1,2,4,5)) \Rightarrow c(0) = 3,\ c(1) = 2.$$

[Definition]

For a group Γ, a bijection $\phi : \Gamma \to \Gamma$ is called skew-morphism with power function $\pi : \Gamma \to \mathbb{Z}$ if it holds that $\phi(1_\Gamma) = 1_\Gamma$ and $\phi(gh) = \phi(g)\phi^{\pi(g)}(h)$ for all $g, h \in \Gamma$.

[Lemma]

A Cayley map $CM(\Gamma : X, p)$ is regular $\Leftrightarrow$ $|\mathrm{Aut}^{+}(\mathfrak{M})_{1_\Gamma}| = |X|$

$\Leftrightarrow$ $\exists$ a skew-morphism $\phi : \Gamma \to \Gamma$ such that $\phi(X) = X$ and $\phi|_X = p$.

[Note]

1. For a given group Γ, the classification of regular maps over Γ is equivalent to classification of skew-morphisms of Γ and their orbits X such that $X^{-1} = X$ and $\Gamma = \langle X \rangle$.

2. $1_\Gamma = \phi(1_\Gamma) = \phi(x_i x_{c(i)}) = \phi(x_i)\phi^{\pi(x_i)}(x_{c(i)}) = x_{i+1} x_{c(i)+\pi(x_i)}$

$\Rightarrow$ $\pi(x_i) = c(i+1) - c(i)$ for all $x_i \in X$.

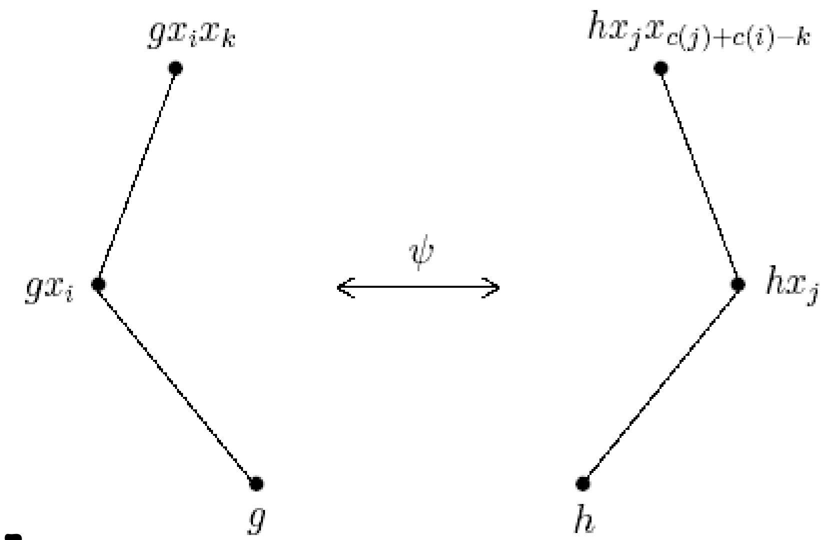
3. $CM(\Gamma : X, p)$: balanced $\Rightarrow$ $\pi(g) = 1$ for all $g \in \Gamma$.

4. $CM(\Gamma : X, p)$: anti balanced (t-balanced,resp.) $\Rightarrow$

$\pi(g) = 1$ if $g \in \Gamma^{+}$ and $\pi(g) = -1$ (t, resp.) if $g \in \Gamma^{-}$.

[Lemma]

Let $\mathfrak{M} = CM(\Gamma, X, p = (x_0, x_1, \ldots, x_{d-1}))$ be a reflexible Cayley map, and let ψ be an orientation-reversing automorphism of $\mathfrak{M}$. If ψ takes the arc (g, gx_i) to the arc (h, hx_j), then $\psi(gx_i x_k) = hx_j x_{c(j)+c(i)-k}$ for all $x_k \in X$.



[Some Results]

1. Cyclic groups (11 M. Conder and T. Tucker)
2. t-balanced for dihedral, dicyclic,..(Kwak, K, R. Feng, Oh,..)

Reflexible Cayley maps on dihedral groups

[Definition]

1. $D_n = \langle a,\ b \mid a^n = b^2 = (ab)^2 = 1 \rangle$: dihedral group of order 2n.
   Let $A = \langle a \rangle$ and $B = D_n - A$.

2. Let a^i be an A-type element and $a^i b$ a B-type element.

Note that a Cayley map CM(D_n: X, p) is balanced $\Leftrightarrow$ all elements in X are B-type element.

3. A Cayley map CM(D_n: X, p) is called alternating if p sends A-type(resp. B-type) element to B-type(resp. A-type) element.

[Main Theorem]

Any reflexible regular Cayley map $CM(D_n : X, p)$ is isomorphic to one of the following maps:

(In fact, all maps in the following are reflexible regular)

1. (balanced) $p=(b\ \ ab\ \ a^{\ell+1}b\ \ a^{\ell^2+\ell+1}b\ \cdots\ a^{\ell^{d-2}+\cdots+\ell+1}b)$, where $\ell^{d-1}+\cdots+\ell+1=0$ (mod n) and $\ell^2=1$ (mod n).
2. (alternating) $p=(a\ \ b\ \ a^{-1}\ \ a^{-2}b\ \ a^{-3}\ \cdots\ a^3\ \ a^2b)$ for any even n

   $p=(a\ \ b\ \ a^{-1}\ \ a^{-2+\frac{n}{2}}b\ \ a^{-3}\ \ a^{-4}b\cdots\ a^3\ \ a^{2+\frac{n}{2}}b)$ for $n=8k$.

3. $p=(a\ \ a^{-1}\ \ b\ \ a^{-2}b)$, where $n=3k$.
4. $p=(a\ \ a^{-1}\ \ b\ \ a^{\frac{n}{2}-1}\ \ a^{\frac{n}{2}+1}\ \ a^{\frac{n}{2}-2}b)$, where $n=8k+4$.
5. $p=(a\ \ b\ \ a^{\frac{n}{2}+2}b\ \ a^{-1}\ \ a^2b\ \ a^{\frac{n}{2}}b)$, where $n=4k+2$.

[Sketch of proof]

Balanced Case. $p = (b \;\; ab \;\; a^{\ell+1}b \;\; a^{\ell^2+\ell+1}b \;\; \cdots a^{\ell^{d-2}+\cdots+\ell+1}b \;)$

ψ : reflection such that $\psi(1) = 1, \;\psi(b) = b.$

$\Rightarrow \psi$: group auto s.t. $\psi(a) = a^{\ell^{d-2}+\cdots+\ell+1}$ and $\psi(b) = b.$

Let $r = \ell^{d-2} + \cdots + \ell + 1$. Then $r^2 = 1$ and $\ell r + 1 = 0.$

$\Rightarrow \;\; \ell = -r$ and $\ell^2 = 1$.

Other Case. $p = (x_0 \;\; x_1 \;\; x_2 \;\; \cdots x_{d-1})$

Assume that x_0 : $A-$type and let ψ be a reflection

such that $\psi(1) = 1, \;\psi(x_0) = x_{c(0)}.$

Note that for any $a^i \in \langle x_0 \rangle, \;\psi(a^i) = a^{-i}$ and $\psi(x_k) = x_{c(0)-k}.$

1. ψ sends A-type(resp. B-type) to A-type(B-type).

Assume that $\exists$ A-type element x_k s.t. $x_{c(0)-k}$ is B-type.

$$x_{c(0)-k}=\psi(x_k)=\psi(x_0 x_k x_{c(0)})=x_{c(0)} x_{c(0)-k} x_{c(0)-k+c(k)-c(0)}$$
$$=x_{c(0)} x_{c(0)-k} x_{c(k)-k}=x_{c(0)-k} x_0 x_{c(k)-k}.$$

$\Rightarrow$ $x_{c(k)-k}=x_{c(0)}$.

$\Rightarrow$ $\exists$ ref. ψ_1 s.t. $\psi_1(x_0)=x_{c(k)}$ and $\psi_1(x_k)=x_{c(0)}$.

$\Rightarrow$ ψ_1 restricted to subgroup $\langle x_0, x_k\rangle$ is group auto.

$\Rightarrow$ $\langle x_0\rangle=\langle x_k\rangle$ and $x_{c(0)-k}=\psi(x_k)=x_k^{-1}=x_{c(k)}$, a contradiction.

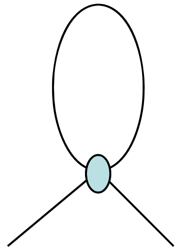
2. ψ is a group automorphism of D_n.

3. ψ_2 is a reflection s. t. $\psi_2(x_k)=x_{c(k)}$ for some A-ele. x_k

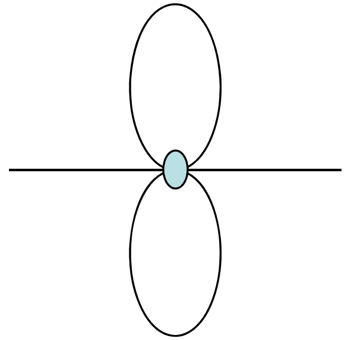
$\Rightarrow$ $\psi=\psi_2$.

Let $\alpha = \min\{|c(k) - k| : x_k$ is A-type ele. in X $\}$.

Case 1 $\alpha = 1$. All possible local structures.

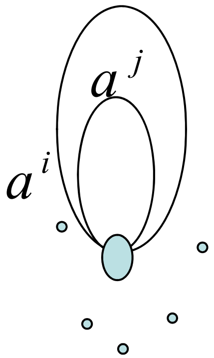


$p = (a \quad a^{-1} \quad b \quad a^{-2}b)$, where $n = 3k$.



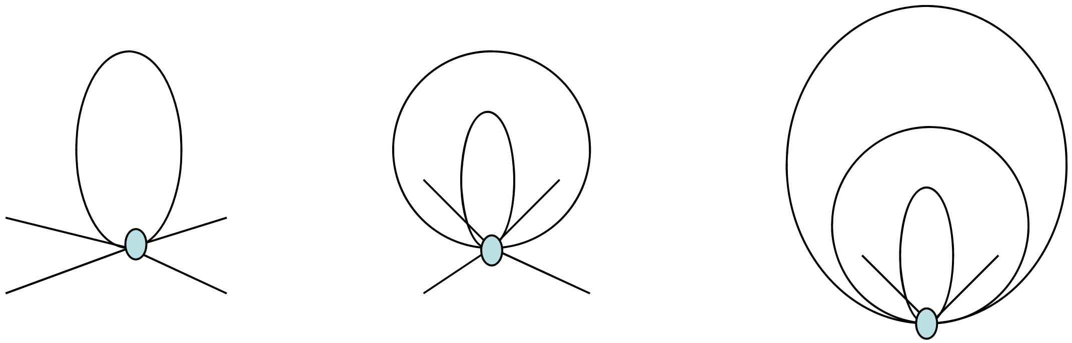
$p = (a \quad a^{-1} \quad b \quad a^{\frac{n}{2}-1} \quad a^{\frac{n}{2}+1} \quad a^{\frac{n}{2}-2}b)$, where $n = 8k + 4$.

Why. Forbidden local structures.



$\varphi(a^{i+j}) = a^{i+j}$ and $\varphi(a^{j+i}) = a^{-j}p^{-1}(a^i)$

$\Rightarrow$ $p^{-1}(a^i)$ is A-type. Similarly, one can show that all elements in X are A-type, a contradiction.



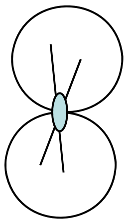
Case 2 $\alpha=2$. All possible local structures.

Alternating structures!!

$$p = (a\ \ b\ \ a^{-1}\ \ a^{-2}b\ \ \ a^{-3}\ \ \cdots\ \ a^{3}\ \ \ a^{2}b)$$ for any even n

$$p = (a\ \ b\ \ a^{-1}\ \ a^{-2+\frac{n}{2}}b\ \ \ a^{-3}\ \ a^{-4}b\cdots\ \ a^{3}\ \ \ a^{2+\frac{n}{2}}b)$$ for $n = 8k$.

Case 3 $\alpha=3$. All possible local structures.



$$p = (a\ \ \ \ b\ \ \ \ a^{\frac{n}{2}+2}b\ \ \ \ a^{-1}\ \ \ a^{2}b\ \ \ a^{\frac{n}{2}}b),$$

where $n = 4k + 2$.

Case 4 $\alpha \geq 4$. No admissible local structure exists.

Regular Cayley maps of odd valency

[Main Theorem]

If $\mathfrak{M} = CM(D_n, X, p)$ is a regular Cayley map of ▷ odd valency k, then $\mathfrak{M}$ is balanced or isomorphic to $CM(D_4, X, p = (a\ \ a^{-1}\ \ b))$.

[Sketch of proof]

1. I. Istvan, D. Marusic and M. Muzychuk classified regular Cayley graphs Γ on D_n s.t. a group G of automorphisms of Γ acts regularly on arcs and A_L is core-free in G.

(1) n=2, $\Gamma=K_4$ and $G \cong A_4$.

(2) n=3, $\Gamma=K_{2,2,2}$ and $G \cong S_4$.

(3) n=4, $\Gamma=Q_3$ and $G \cong S_4$.

(4) n=2m with odd m, $\Gamma=K_{n,n}$ and $G \cong (D_n \times D_n) \rtimes \langle \alpha \rangle$

$(x,y)^{\alpha} = (y,x)$.

(1) and (3) are odd valent.

2. Assume that $\mathfrak{M} = CM(D_n$:X,p) is a regular non-balanced Cayley map of odd valency such that A_L is not core free and its order is possibly smallest.

$\Rightarrow$ $\exists\, C_q \lhd$ G with prime q such that $\mathfrak{M} / C_q$ is isomorphic to regular Cayley corresponding to (1) and (3). $\Rightarrow$ *no* such $\mathfrak{M}$ exists.

Future work

1. Classification of regular Cayley map on dihedral group.

2. Classification of t-balanced regular Cayley map on abelian group.

3. Classification of reflexible regular Cayley maps on abelian groups.

4. ……

Thank you!!!!