Chord Properties in Euclidean Geometry

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Abstract

- **Abstract**

  First, we study curves in a Euclidean space of arbitrary dimension such that the chord joining any two points on the curve meets it at the same angle.

  Next, we study hypersurfaces in a Euclidean space of arbitrary dimension such that the chord joining any two points on the hypersurface meets it at the same angle.

  As a result, we give a complete characterization of such curves (hypersurfaces, resp.) in Euclidean space $\mathbb{E}^m$ with arbitrary dimension.

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Introduction

It is well-known that a circle is characterized as a closed plane curve such that the chord joining any two points on it meets the curve at the same angle at the two points (cf. [8, pp. 160-162]).

From differential geometric point of view, this characteristic property of circles can be stated as follows:

Proposition 1. Let $X = X(s)$ be a unit speed closed curve in the Euclidean plane $\mathbb{E}^2$ and $T(s) = X'(s)$ be its unit tangent vector field. Then $X = X(s)$ is a circle if and only if it satisfies the following chord property:

(C): $\langle X(t) - X(s), T(t) - T(s)\rangle = 0$ holds identically.

Introduction

Actually, one can show the following:

Proposition 2. A unit speed plane curve $X(s)$ satisfies chord property (C) if and only if it is either a circle or a straight line.

In views of above propositions, it is natural to ask the following question:

"Which Euclidean space curves satisfy the chord property (C)?"

Introduction

For a sphere $S^2(r)$ in a 3-dimensional Euclidean space $\mathbb{E}^3$, the chord joining any two points on it meets the sphere at the same angle at the two points, that is, the sphere satisfies the chord property:

(D): $\langle y - x, G(x) + G(y) \rangle = 0$ holds identically, where G denotes the Gauss map.

Hence, it is also natural to ask the following question:

"Which hypersurfaces in an m-dimensional Euclidean space $\mathbb{E}^m$ satisfy the chord property (D)?"

W-curves

A curve in a Euclidean space is called a W-curve if its Frenet curvatures are constant. Straight lines, circles and circular helices in $\mathbb{E}^3$ are the simplest examples of W-curves ([6]).

With respect to a suitable Euclidean coordinate system of $\mathbb{E}^m$, every unit speed W-curve $X(s)$ in $\mathbb{E}^m$ can be written as follows:

$$X(s) = (a_1 \cos c_1 s, a_1 \sin c_1 s, \cdots, a_n \cos c_n s, a_n \sin c_n s, 0, \ldots, 0)$$

or as

$$X(s) = (a_1 \cos c_1 s, a_1 \sin c_1 s, \cdots, a_n \cos c_n s, a_n \sin c_n s, bs, 0, \ldots, 0)$$

for some distinct nonzero numbers $c_1, \ldots, c_n$ and a nonzero number b.

W-curves

First of all, we may get the following:

Theorem A. (Chen, Kim and Kim, [3]) For a unit speed curve $X(s)$ in $\mathbb{E}^m$, TFAE:

(i) $X(s)$ satisfies chord property (C).

(ii) $|X(s+a) - X(s)|$ depends only on a.

(iii) $\langle X^{(i)}(s), X^{(j)}(s) \rangle$, $i + j = 2, \cdots, 2m$, are constant.
(iv) $|X^{(k)}(s)|$, $k = 1, \cdots, m$, are constant.

(v) $X(s)$ is a W-curve.

isoparametric hypersurfaces

A hypersurface in a Euclidean space is called an isoparametric hypersurface if its principal curvatures are constant.

Planes, spheres and circular cylinders in $\mathbb{E}^3$ are the simplest examples of isoparametric hypersurfaces.

isoparametric hypersurfaces

For Euclidean hypersurfaces satisfying chord property (D), we have the following:

Theorem B. (Kim and Kim, [5]) For a hypersurface M in Euclidean m-space $\mathbb{E}^m$, the following are equivalent:

(i) M satisfies chord property (D).

(ii) For an $m \times m$ matrix A and a vector $b \in \mathbb{E}^m$, we have

$$G(x) = Ax + b.$$

(iii) M is an isoparametric hypersurface.

(iv) M is an open part of one of the following hypersurfaces:

$$\mathbb{E}^{m-1}, S^{m-1}(r), S^{p-1}(r) \times \mathbb{E}^{m-p}.$$

Proof of Theorem A

Let $X = X(s)$ be a unit speed smooth curve in Euclidean m-space. Without loss of generality, we may assume that $X = X(s)$ is defined on an open interval I containing 0.
Suppose that the curve satisfies
chord property (C): $\langle X(t) - X(s), T(t) - T(s)\rangle = 0$.
Then, by putting $t = s + a$, we obtain

$$\langle X(s+a) - X(s), T(s+a) - T(s)\rangle = 0. \tag{2.1}$$

It follows from equation (2.1) that

$$|X(s+a) - X(s)|^2 = f(a) \tag{2.2}$$

for some function $f = f(a)$. From (2.2) we find

$$\begin{aligned} f(-a) &= |X(s-a) - X(s)|^2 \\ &= |X(s-a+a) - X(s-a)|^2 = f(a), \end{aligned} \tag{2.3}$$

which implies that $f(a)$ is an even function.

Proof of Theorem A

Let us consider Taylor's expansion of $f(a)$ about $a = 0$. Since $f(a)$ is an even function, we have

$$f(a) = \sum_{k=2}^{2m} c_k a^k + O(|a|^{2m+1}) \quad \text{as} \quad a \to 0, \tag{2.4}$$

for some constants $c_2, \dots, c_{2m}$, where $O(|a|^{2m+1})$ is a function $g(a)$ satisfying $|g(a)| \leqslant C|a|^{2m+1}$ for some constant C and sufficiently small $a > 0$. Let us also consider Taylor's expansion of $X(s+a)$ about $a = 0$ which enable

$$X(s+a) - X(s) = \sum_{k=1}^{2m-1} \frac{1}{k!} X^{(k)}(s) a^k + O(|a|^{2m}). \tag{2.5}$$

Proof of Theorem A

From (2.2) and (2.5) we find

$$f(a) = \sum_{k=2}^{2m} \left(\sum_{i=1}^{k-1} \frac{1}{i!(k-i)!} \langle X^{(i)}(s), X^{(k-i)}(s) \rangle \right) a^k + O(|a|^{2m+1}) \tag{2.6}$$

as $a \to 0$.

Hence we obtain

$$c_k = \sum_{i=1}^{k-1} \frac{1}{i!(k-i)!} \langle X^{(i)}(s), X^{(k-i)}(s) \rangle \tag{2.7}$$

for $k = 2, \cdots, 2m$.

Now, we may prove by mathematical induction that

$$\langle X^{(i)}(s), X^{(k-i)}(s) \rangle \text{ is constant for } i = 1, \cdots, k-1; \;\; 2 \leqslant k \leqslant 2m. \tag{2.8}$$

Proof of Theorem A

Then, we may get Theorem A. For details, see [3].

Theorem A. (Chen, Kim and Kim, [3]) For a unit speed curve $X(s)$ in $\mathbb{E}^m$, TFAE:

(i) $X(s)$ satisfies chord property (C).

(ii) $|X(s+a) - X(s)|$ depends only on a.

(iii) $\langle X^{(i)}(s), X^{(j)}(s) \rangle$, $i + j = 2, \cdots, 2m$, are constant.
(iv) $|X^{(k)}(s)|$, $k = 1, \cdots, m$, are constant.

(v) $X(s)$ is a W-curve.

Proof of Theorem B

Let M be a hypersurface in a Euclidean space $\mathbb{E}^m$ which satisfy

chord property (D): $\langle y - x, G(x) + G(y)\rangle = 0$

or

chord property (D): $\langle G(x), y\rangle = \langle G(x), x\rangle + \langle G(y), x\rangle - \langle G(y), y\rangle$.

Without loss of generality, we may assume that M is not contained in any hyperplane, that is, M is full in $\mathbb{E}^m$.

Then on M, there exist points $y_0, y_1, \cdots, y_m$ such that the set $\{y_j - y_0 | j = 1, 2, \cdots, m\}$ spans the Euclidean space $\mathbb{E}^m$.

From chord property (D) we have for $j = 1, 2, \cdots, m$

$$\langle G(x), y_0\rangle = \langle G(x), x\rangle + \langle G(y_0), x\rangle - \langle G(y_0), y_0\rangle, \tag{3.1}$$

$$\left\langle G(x), y_j\right\rangle = \langle G(x), x\rangle + \left\langle G(y_j), x\right\rangle - \left\langle G(y_j), y_j\right\rangle. \tag{3.2}$$

Proof of Theorem B

By subtracting (3.1) from (3.2), we obtain

$$\langle G(x), A_j \rangle = \langle B_j, x \rangle + c_j, j = 1, 2, \cdots, m, \tag{3.3}$$

where we put

$$A_j = y_j - y_0, B_j = G(y_j) - G(y_0), c_j = \langle G(y_0), y_0 \rangle - \langle G(y_j), y_j \rangle$$

for $j = 1, 2, \cdots, m$.
Hence we may prove the following:

Lemma 3.1. For an $m \times m$ matrix A and a vector $b \in \mathbb{E}^m$ we have $G(x) = Ax + b$.

Proof of Theorem B

By differentiating G covariantly with respect to a tangent vector X to M, it follows from Lemma 3.1 that

$$AX = -S(X), X \in T_x M, \tag{3.4}$$

where S denotes the shape operator. Choose an orthonormal frame $E_1, \cdots, E_{m-1}$ such that $E_1, \cdots, E_{m-1}$ are eigenvectors of S with eigenvalues $\mu_1, \cdots, \mu_{m-1}$. Then from (3.4), for all $x \in M$ we have

$$AE_j(x) = -\mu_j(x)E_j(x), j = 1, 2, \cdots, m-1. \tag{3.5}$$

Since A is a constant matrix and the set of eigenvalues of a matrix is discrete, the principal curvatures $\mu_1, \cdots, \mu_{m-1}$ are all constant, that is, M is an isoparametric hypersurface ([4]). Hence it follows from a well-known theorem([7, 9]) that M is an open part of either a sphere $S^{m-1}(r)$ or a generalized cylinder $S^{p-1}(r) \times \mathbb{E}^{m-p}$.

Proof of Theorem B

Here, we give an elementary proof. Since S is self-adjoint and $\{X \in T_x M | x \in M\}$ spans $\mathbb{E}^m$, (3.4) shows that A is symmetric.

For the function $f : \mathbb{E}^m \to R$ defined by $f(x) = \langle Ax + b, Ax + b \rangle$, it follows from Lemma 3.1 that $M \subset f^{-1}(1)$ because $G(x)$ is a unit vector field.

This shows that the gradient vector $\nabla f(x) = 2A(Ax + b)$ is proportional to $G(x)$. Hence for some function $\lambda(x)$ we have

$$A(Ax + b) = \lambda(x)(Ax + b), x \in M. \tag{3.6}$$

Since the eigenvalues of a matrix form a discrete set, $\lambda(x)$ must be a constant. It follows from (3.6) that $V = \text{Span}\{Ax + b | x \in M\}$ is contained in an eigenspace of A corresponding to eigenvalue λ.

Proof of Theorem B

From the assumption that M is not contained in any hyperplane, as in the proof of Lemma 3.1 we see that

$$\mathrm{Im}A = \mathrm{Span}\{AA_j | j = 1, 2, \cdots, m\} \subset V. \tag{3.7}$$

It follows from (3.6) that

$$(A^2 - \lambda A)x = -Ab + \lambda b, x \in M. \tag{3.8}$$

Hence we have

$$(A^2 - \lambda A)A_j = 0, j = 1, 2, \cdots, m, \tag{3.9}$$

which shows

$$A^2 - \lambda A = 0. \tag{3.10}$$

Proof of Theorem B

Suppose that $\lambda = 0$. Then (3.10) shows that $A^2 = 0$. Since A is symmetric, A must vanish. Together with Lemma 3.1, this shows that M is an open part of an hyperplane $\mathbb{E}^{m-1}$.

This contradiction implies that $\lambda \neq 0$. Together with (3.10), (3.8) shows that $b = \frac{1}{\lambda}Ab \in \mathrm{Im}A$. Hence (3.7) implies $V = \mathrm{Im}A$.

Let's denote $\lambda = \pm\frac{1}{r}$ with $r > 0$. Then we have $A|_V = \pm\frac{1}{r}I$. We may prove Theorem B according to the dimension of V. For details, see [5].

Proof of Theorem B

Hence for Euclidean hypersurfaces satisfying chord property (D), we have the following:

Theorem B. (Kim and Kim, [5]) For a hypersurface M in Euclidean m-space $\mathbb{E}^m$, the following are equivalent:

(i) M satisfies chord property (D).

(ii) For an $m \times m$ matrix A and a vector $b \in \mathbb{E}^m$, we have

$$G(x) = Ax + b.$$

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(iv) M is an open part of one of the following hypersurfaces:

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References I

- H. P. Boas, *A geometric characterization of the ball and the Bochner-Martinelli kernel*, Math. Ann. 248(1980), 275-278.
- H. P. Boas, *Spheres and cylinders; A local geometric characterization*, Illinois J. Math. 28/1(1984), 120-124.
- B.-Y. Chen, D.-S. Kim and Y.H. Kim, *New characterization of W-curves*, Publ. Math. Debrecen 69/4(2006), 457-472.
- D.-S. Kim, *On the Gauss map of hypersurfaces in the space form*, J. Korean Math. Soc. 32/3(1995), 509-518.
- D.-S. Kim and Y.H. Kim, *New characterizations of spheres, cylinders and W-curves*, Linear Algebra Appl. 432 (2010), no. 11, 3002-3006.

References II

- W. Kühnel, *Differential geometry, curves-surfaces-manifolds*, Translated from the 1999 German original by Bruce Hunt, Student Mathematical Library 16, American Mathematical Society, Providence, RI, 2002.
- T. Levi-Civita, *Famiglie di superficie isoparametriche nell'ordinario spacio euclideo*, Rend. Acad. Lincei 26(1937), 355-362.
- H. Rademacher and O. Toeplitz, *The enjoyment of mathematics*, Translated from the second (1933) German edition and with additional chapters by H. Zuckerman. Princeton Science Library, Princeton University Press, Princeton, NJ, 1994.

References III

- B. Segre, *Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di demensioni*, Rend. Acad. Lincei 27(1938), 203-207.
- B. Wegner, *A differential geometric proof of the local geometric characterization of spheres and cylinders by Boas*, Mathematica Balkanica, New Series 2/4(1988), 294-295.

Thank You!