Highly symmetric hyperbolic polytopes

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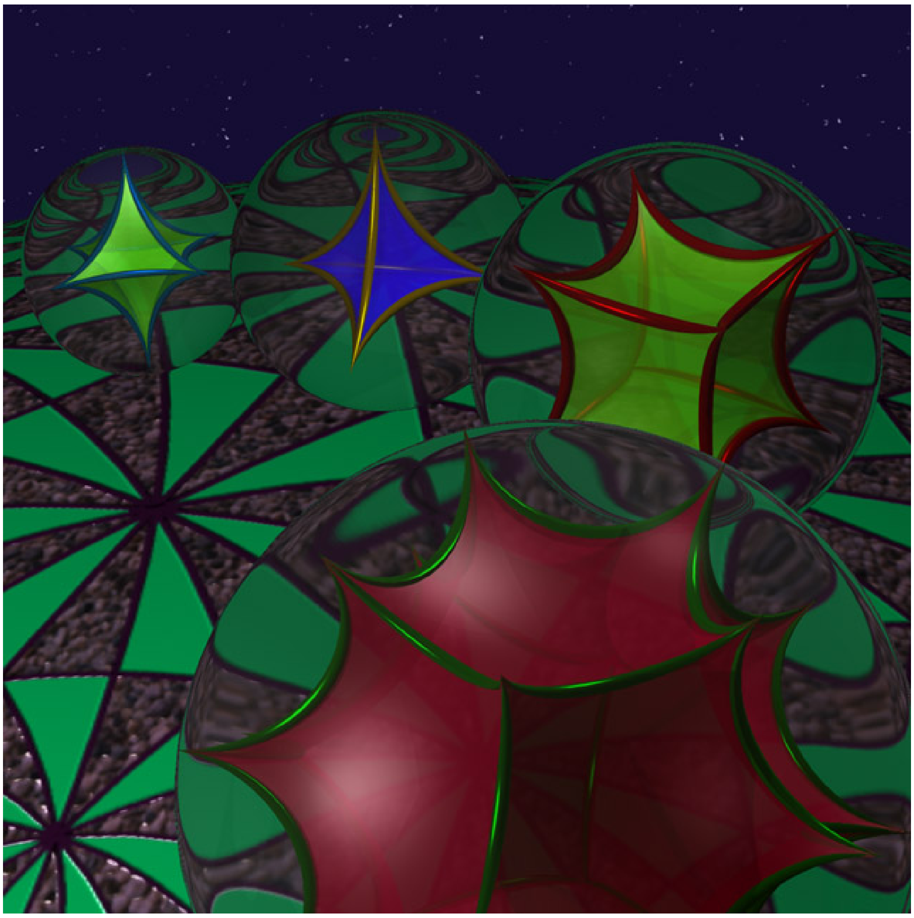


Aims :

- symmetric polytopes, providing "periodic" tesselations
- discrete group actions of small (minimal) covolume or
  top down ranking w.r.t. symmetry degree
- describe related combinatorial and number theoretical features

Hyperbolic polyhedral geometry

Hyperbolic space in the Lorentz-Minkowski space :

$$\mathbb{R}^{n,1} \quad = (\, \mathbb{R}^{n+1}, < x, y >= x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1} \,)$$

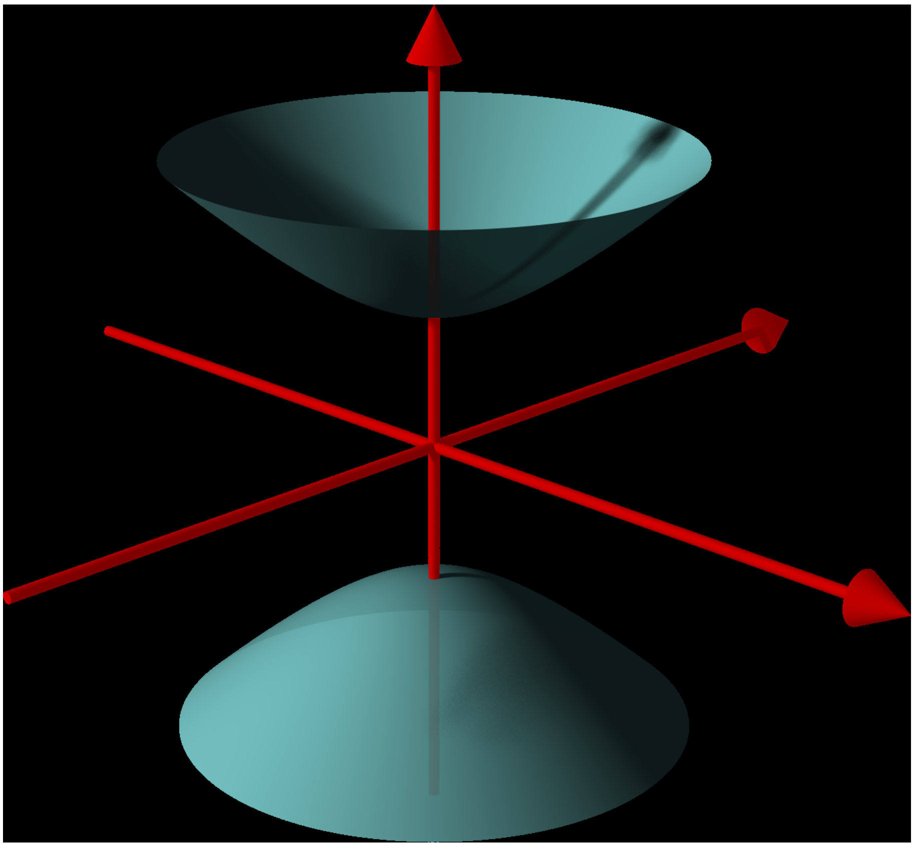
$$\mathbb{H}^n \quad = \{x \in \mathbb{R}^{n,1} \mid |x|^2 =< x, x >= -1 \,,\, x_{n+1} > 0\}$$

Geodesic k-planes U in $\mathbb{H}^n$ $(1 \le k \le n-1)$:

$$U \quad = V^{k+1} \cap \mathbb{H}^n$$

$$H \quad = V^n \cap \mathbb{H}^n = e^{\perp} \quad \text{for} \quad e \in \mathbb{R}^{n,1}\,,\, |e| = 1 \quad \text{with}$$

$$H^- \quad := \{x \in \mathbb{H}^n \mid < x, e > \le 0\}$$



$$\mathrm{Isom}(\mathbb{H}^n) \cong PO_0(n,1) = \{A \in GL(n+1;\mathbb{R}) \mid AJA^t = I_{n+1}\,,\, [A]_{n+1} > 0\}$$

$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$; every $A \in \mathrm{Isom}(\mathbb{H}^n)$ is a finite composition of reflections w.r.t. hyperplanes H

Convex polytopes in hyperbolic space

Convex polytope in $\mathbb{H}^n$: $\quad P = \cap_{i\in I} H_i^-\ ,\ H_i = e_i^{\perp}\ , \mid e_i \mid = 1,$ with

Gram matrix $G(P) = (< e_i, e_j >)$

$$- < e_i, e_j > = \begin{cases} \cos\alpha_{ij}, & \angle(H_i, H_j) \quad = \alpha_{ij} \\ \cosh l_{ij}, & \mathrm{dist}(H_i, H_j) \quad = l_{ij} \end{cases}$$

- **Vinberg's realisation criterion**

Given an indecomposable symmetric matrix $G = (g_{ij})$ of signature $(n, 1)$ with diagonal entries equal to 1 and $g_{ij} \leq 0$ otherwise. Then, there is an acute-angled convex polytope $P \subset \mathbb{H}^n$ with $G(P) = G$; it is unique up to an isometry

- **combinatorial and metrical structure of P**

Information about the face complex, the number of ideal and ultra-ideal vertices, compactness, finite volume for an acute-angled polytope P is given by $G(P)$ and the types of its submatrices G_J

E.g. P compact simplex if G has order $n+1$, signature $(n, 1)$ and each principal submatrix G_J is positive definite

- **Fundamental polytopes for discrete groups in Isom$(\mathbb{H}^n)$**

Of special interest are

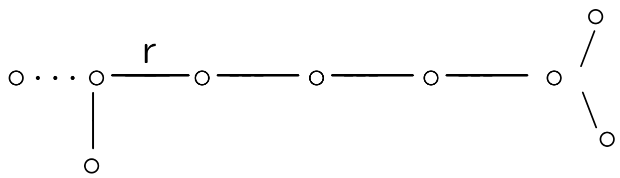
○ Polytopes with dihedral angles satisfying Poincaré's polyhedron theorem and giving rise to discrete groups (difficult to construct ...)

○ Polytopes with dihedral angles π/k for integers $k \geq 2$: They are acute-angled and fundamental polytopes for groups generated by reflections w.r.t. finitely many hyperplanes

$\longrightarrow$ **Coxeter polytopes and Coxeter groups**

Coxeter graphs and Coxeter polytopes

A *geometric Coxeter graph* Σ is a weighted graph of the form



giving rise to a non-Euclidean Coxeter n-polytope if the associated symmetric Gram matrix is positive definite or of signature $(n,1)$

○	hyperplane H (or mirror)
○ —r— ○ (i, k)	$\angle(H_i, H_k) = \frac{\pi}{r}$ for $2 < r \leq \infty$
○ — ○	$\angle(H_i, H_k) = \frac{\pi}{3}$
○ ○	$H_i \perp H_k$
○ · · · · · ○ (l)	$\mathrm{dist}_{\mathbb{H}}(H_i, H_k) = l$

Important and simplest examples

- Linear graphs of order $n+1$ are **Coxeter orthoschemes** in X^n_K

for $K = +1$: $A_n, B_n, D_n, E_6, E_7, E_8, H_3, H_4, G_p^2$ finite (spherical)

for $K = -1$: Gram(Σ) of signature $(n,1)$

- The graph with 7 nodes realises a truncated simplex in $\mathbb{H}^5$

○ —r— ○ — ○ — ○ — ○ — ○ · · · ○ , $\quad r = 4, 5$

Cocompact hyperbolic Coxeter orthoschemes

n=2 :

$$\circ\overset{p}{\text{———}}\circ\overset{q}{\text{———}}\circ \quad , \quad \frac{1}{p}+\frac{1}{q}<\frac{1}{2}$$

n=3 :

$$\circ\text{———}\circ\overset{5}{\text{———}}\circ\text{———}\circ \qquad \circ\overset{4}{\text{———}}\circ\text{———}\circ\overset{5}{\text{———}}\circ \qquad \circ\overset{5}{\text{———}}\circ\text{———}\circ\overset{5}{\text{———}}\circ$$

n=4 :

$$\circ\text{———}\circ\text{———}\circ\text{———}\circ\overset{5}{\text{———}}\circ$$

$$\circ\overset{4}{\text{———}}\circ\text{———}\circ\text{———}\circ\overset{5}{\text{———}}\circ \qquad \circ\overset{5}{\text{———}}\circ\text{———}\circ\text{———}\circ\overset{5}{\text{———}}\circ$$

Vinberg's arithmeticity criterion

Let $G = (g_{ij})$ be the Gram matrix of a cofinite hyperbolic Coxeter group Γ (and its fundamental polytope P) in $\mathbb{H}^n$.

Let F be the field generated by all cycles $g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_{k-1} i_k} g_{i_k i_1}$, and let $\widetilde{F}$ be the field generated by all entries of G.

The group Γ *is arithmetic (and defined over F) if and only if*

(1) $\widetilde{F}$ is totally real

(2) for any embedding $\sigma : \widetilde{F} \to \mathbb{R}$ with $\sigma|_F \neq id$:

the matrix $G^\sigma := (g^\sigma_{ij})$ *is positive semi-definite*

(3) the cyclic products of the matrix $2G$ are integers of F.

Remarks.

- *Condition (2) is equivalent to*

(2)' all principal minors (being elements in F) are non-negative;

(2)" for any non-trivial embedding $\sigma : F \to \mathbb{R}$:

the matrix $G^\sigma = (g^\sigma_{ij})$ *has non-negative principal minors ;*

in this case, condition (1) is automatically verified;

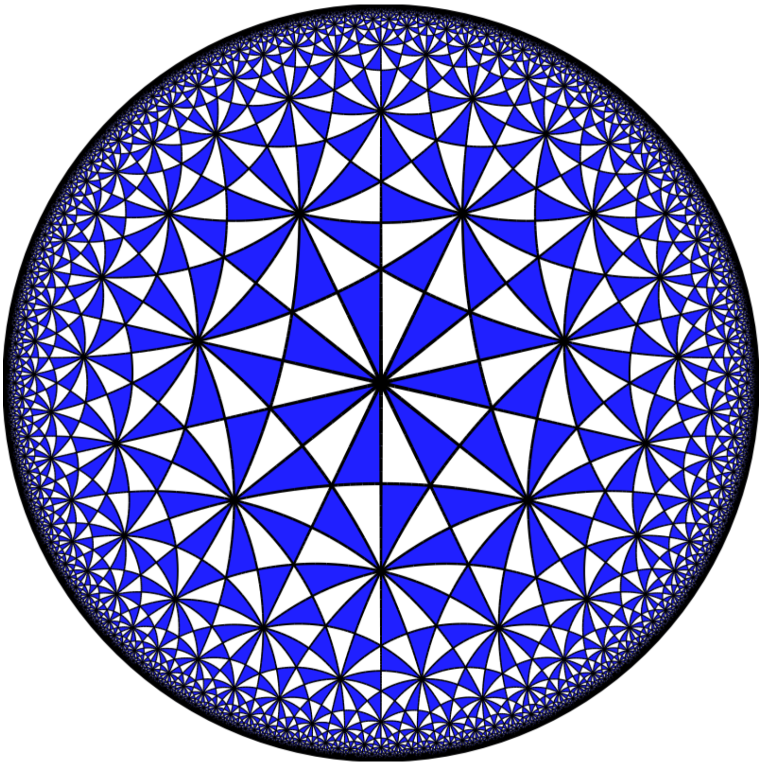
- *Condition (3) holds trivially if the Coxeter graph of* Γ *contains* **no dotted** *edges. In fact, the non-diagonal entries are then related to algebraic numbers of the form* $2 \cos \pi/r$ *.*

- *If the polytope P is* **non-compact but of finite volume**, *then the arithmeticity of* Γ *(over the field $\mathbb{Q}$) is equivalent to the condition that*

all cyclic products of the matrix $2G$ are rational integers

Discrete groups acting cocompactly with minimal hyperbolic covolume

n=2 :



The triangle Coxeter group $\circ\!-\!\!\!-\!\circ\overset{7}{-\!\!\!-}\circ$ *realising the (unique) discrete group acting with minimal coarea* $\pi/42$ *on* $\mathbb{H}^2$*; it is arithmetic* (Siegel, 1945)

Recall: For an oriented compact Riemannian surface $S_g = \mathbb{H}^2/\Gamma$ with fundamental domain P for Γ:

$$\mathrm{vol}(S_g) = 4\pi(g-1)$$

For the hyperbolic 2-orbifold $Q = S_g/\mathrm{Aut}(S_g)$:

$$\mathrm{ord}(\mathrm{Aut}(S_g)) = \frac{\mathrm{vol}(S_g)}{\mathrm{vol}(P)} \leq \frac{4\pi(g-1)}{\pi/21} = 84(g-1) \quad \text{(Hurwitz, 1893)}$$

Bound is sharp for Klein's quartic given by $x^3y + y^3z + z^3x = 0$ *of genus 3*

n=3 :

Look for polyhedra inducing tesselations of $\mathbb{H}^3$ and being of simple combinatorial type with "large" dihedral angles

Check e.g. all cocompact hyperbolic Coxeter groups with few generators and small weights...

- Candidate is the (arithmetic) Coxeter group

$$\circ\overset{5}{\text{——}}\circ\text{——}\circ\overset{4}{\text{——}}\circ \quad \text{of covolume} \simeq 0.03589$$

associated to a right-angled regular dodecahedron $\{5,3\}$

- T. Chinburg-E. Friedmann, 1986 :

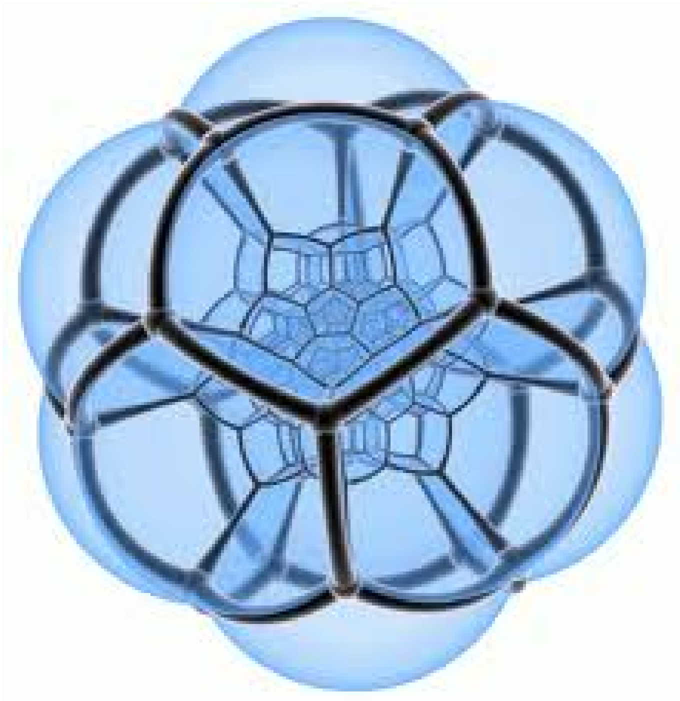
$$\tfrac{1}{2}\,\mathrm{vol}_3(\circ\text{——}\circ\overset{5}{\text{——}}\circ\text{——}\circ) \simeq 0.0195 = \tfrac{1}{2}\,0.03905$$

realises the minimal volume of all **arithmetic** hyperbolic 3-orbifolds

- F. Gehring-G. Martin announced a result that this value is in fact minimal for **ALL** hyperbolic 3-orbifolds of finite volume

Part I of the proof is published in Ann. Math., 2009; Part II ?

n=4 :



The hyperbolic 120-cell $\{5,3,3\}$ *with symmetry group generated by the (arithmetic) Coxeter group* Γ : $\circ\overset{5}{—}\circ—\circ—\circ—\circ$

Passing to its rotational subgroup Γ', the space $\mathbb{H}^4/\Gamma'$ is an oriented compact arithmetic 4-orbifold. Now :

Theorem (M. Belolipetsky, 2004)

Let **n** $>$ **2 even**. *Among all orientable compact arithmetic* n-*orbifolds there is precisely one of minimal volume; it is defined over* $\mathbb{Q}(\sqrt{5})$. *For* $n = 4$ *this is* Γ' *which has covolume* $\pi^2/5400$

Minimal volume of arithmetic hyperbolic orbifolds
- the case of odd dimensions ≥ 5 -

Let $\Gamma < \mathrm{PO}_0(n,1)$ be discrete and arithmetical w.r.t. the number field k with quotient space $Q = \mathbb{H}^n/\Gamma$.

Theorem (V. Emery, Ph.D. 2009, Fribourg)

For $\mathbf{n = 2r\text{-}1} \geq 5$*, a orientable compact arithmetic hyperbolic* n*-orbifold* Q_0^n *of minimal volume is defined over* $k_0 = \mathbb{Q}(\sqrt{5})$ *and of volume*

$$\mathrm{vol}_n(Q_0^n) \quad = \quad \frac{5^{r^2-r/2}\cdot 11^{r-1/2}\cdot (r-1)!}{2^{2r-1}\pi^r}\, L_{\ell_0/k_0}(r) \prod_{i=1}^{r-1} \frac{(2i-1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i),$$

where ℓ_0 *is the quartic field with a defining polynomial* $x^4 - x^3 + 2x - 1$ *(and of discriminant* -275*), and where* ζ_{k_0} *denotes the Dedekind zeta function and* $L_{\ell_0/k_0} = \zeta_{l_0}/\zeta_{k_0}$ *is the* L*-function corresponding to the quadratic extension* ℓ_0/k_0

- This theorem completes Belolipetsky's results for all $n \geq 4$
- Together with Belolipetsky, Emery proved the **unicity** of the minimal volume orbifold Q_0^n !

Consider the first interesting case Q_0^5 in odd dimensions

Geometric identification of Q_0^5

Theorem (V. Emery–K, 2012)

The orientable double cover of the quotient space of $\mathbb{H}^5$ by the Coxeter 5-prism group

$$\Gamma_* \quad : \quad \circ\overset{5}{\text{—}}\circ\text{—}\circ\text{—}\circ\text{—}\circ\text{—}\circ\cdots\circ$$

is the (unique) orientable compact arithmetic hyperbolic 5-orbifold Q_0^5 of minimal volume

Idea of proof :

(I) By the result of Emery,

$$(\star) \qquad \mathrm{vol}_5(Q_0^5) = \frac{9\sqrt{5}^{15}\sqrt{11}^5}{2^{14}\pi^{15}}\zeta_{k_0}(2)\zeta_{k_0}(4)L_{\ell_0/k_0}(3)$$

$$\cong 0.00153471916863561864669180372 4$$

(II) By the unicity statement in Emery's theorem, we need only to compute the covolume of the Coxeter 5-prism

$$\Gamma_* \quad : \quad \circ\overset{5}{\text{—}}\circ\text{—}\circ\text{—}\circ\text{—}\circ\text{—}\circ\cdots\circ$$

which is a simply truncated Coxeter orthoscheme (one ultra-ideal vertex, cut away by its polar hyperplane). It can be seen as characteristic polytope of the 5-dimensional (truncated) cousin of the 120-cell $\{5,3,3\}$

$\rightsquigarrow$ *a quick excursion into the realm of hyperbolic simplex volume formulae*

Hyperbolic volume in 3 dimensions

Formula of Lobachevsky

For an orthoscheme $R \subset \mathbb{H}^3$ as given by a weighted linear graph of length 4,

$$\begin{aligned}\mathrm{vol}_3(R) = \frac{1}{4}\Big\{ & \text{Л}_2(\alpha+\theta) - \text{Л}_2(\alpha-\theta) + \text{Л}_2(\frac{\pi}{2}+\beta-\theta) + \\ & + \text{Л}_2(\frac{\pi}{2}-\beta-\theta) + \text{Л}_2(\gamma+\theta) - \text{Л}_2(\gamma+\theta)\Big\} \quad ,\end{aligned}$$

$$0 \le \theta = \arctan \frac{\left(\cos^2\beta - \sin^2\alpha\,\sin^2\gamma\right)^{1/2}}{\cos\alpha\cos\gamma} \le \frac{\pi}{2}$$

$$\begin{aligned}\text{Л}_2(x) &= -\int_o^x \log|2\sin t|\,dt = -\int_o^x \log|1-\exp(2it)|\,dt \\ &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin(2kx)}{k^2} = \frac{1}{2}\,\mathrm{Im}\,\mathrm{Li}_2(e^{2ix})\end{aligned}$$

Lobachevsky's function

Example :

$$\mathrm{vol}_3(\circ\overset{6}{\text{——}}\circ\text{——}\circ\text{——}\circ) = \frac{1}{8}\,\text{Л}(\frac{\pi}{3}) \simeq 0.042289$$

Hyperbolic volume in 5 dimensions

Theorem (K, 1992) *For a 5-orthoscheme* R_∞

$$\circ \overset{\alpha}{\text{——}} \circ \overset{\beta}{\text{——}} \circ \overset{\gamma}{\text{——}} \circ \overset{\alpha}{\text{——}} \circ \overset{\beta}{\text{——}} \circ$$

with $\cos^2\alpha + \cos^2\beta + \cos\gamma^2 = 1$, *the volume is given by*

$$\begin{aligned} \mathrm{vol}_5(R_\infty) = \frac{1}{4}\{\text{Л}_3(\alpha) + \text{Л}_3(\beta) - \frac{1}{2}\text{Л}_3(\frac{\pi}{2} - \gamma)\} - \\ -\frac{1}{16}\{\text{Л}_3(\frac{\pi}{2} + \alpha + \beta) + \text{Л}_3(\frac{\pi}{2} - \alpha + \beta)\} + \frac{3}{64}\zeta(3) \end{aligned}$$

with $\text{Л}_3(x) = \frac{1}{4}\zeta(3) - \int_0^x \text{Л}_2(t)dt = \frac{1}{4}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{k^3} = \frac{1}{4}\,\mathrm{Re}\,\mathrm{Li}_3(e^{2ix})$

Examples :

$$\mathrm{vol}_5(\circ \text{——} \circ \overset{4}{\text{——}} \circ \text{——} \circ \text{——} \circ \overset{4}{\text{——}} \circ) = \frac{7\,\zeta(3)}{4608}$$

$$v_1 = \mathrm{vol}_5(\circ \text{——} \circ \overset{5}{\text{——}} \circ \overset{5/2}{\text{——}} \circ \text{——} \circ \overset{5}{\text{——}} \circ) = \frac{1}{144}\{\text{Л}_3(\tfrac{\pi}{5}) + \tfrac{\zeta(3)}{5}\}$$

$$v_2 = \mathrm{vol}_5(\circ \overset{5/2}{\text{——}} \circ \overset{5}{\text{——}} \circ \text{——} \circ \overset{5/2}{\text{——}} \circ \overset{5}{\text{——}} \circ) = \frac{\zeta(3)}{1200}$$

$$v_3 = \mathrm{vol}_5(\circ \overset{5/2}{\text{——}} \circ \text{——} \circ \overset{5}{\text{——}} \circ \overset{5/2}{\text{——}} \circ \overset{5}{\text{——}} \circ) = \frac{1}{144}\{-\text{Л}_3(\tfrac{\pi}{5}) + \tfrac{2\,\zeta(3)}{25}\}$$

Observe : All these polytopes are non-compact !

By using scissors congruence methods (cutting, moving, pasting...)

$$\mathrm{vol}_5(\circ\!-\!\!-\!\circ\overset{4}{-\!\!-\!\!-}\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ) \quad = \quad \tfrac{7\,\zeta(3)}{46080}$$

$$u_1 = \mathrm{vol}_5(\circ\overset{5/2}{-\!\!-\!\!-}\circ\overset{5}{-\!\!-}\circ\overset{5/2}{-\!\!-\!\!-}\circ\overset{5}{-\!\!-}\circ\overset{5/2}{-\!\!-\!\!-}\circ) \quad = \quad \tfrac{1}{96}\,\text{Л}_3(\tfrac{\pi}{5})$$

$$u_2 = \mathrm{vol}_5(\circ\overset{5}{-\!\!-}\circ\overset{5/2}{-\!\!-\!\!-}\circ\overset{5}{-\!\!-}\circ\overset{5/2}{-\!\!-\!\!-}\circ\overset{5}{-\!\!-}\circ) \quad = \quad \tfrac{1}{96}\,\text{Л}_3(\tfrac{\pi}{5}) + \tfrac{\zeta(3)}{800}$$

$$w = \mathrm{vol}_5(\circ\overset{5}{-\!\!-}\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\overset{5/2}{-\!\!-\!\!-}\circ\cdots\circ)$$

$$= \tfrac{1}{20}\,\{3v_1 - 2v_2 + 3v_3\} - \tfrac{1}{12}\{u_1 - u_2\} \quad = \quad \tfrac{\zeta(3)}{3200}$$

On-going work

We are trying to express the covolume of the Coxeter group

$$\Gamma_* \quad : \quad \circ\overset{5}{-\!\!-}\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\;\circ\cdots\circ$$

by scissors congruences in a way as above for w being the covolume of $\circ\overset{5}{-\!\!-}\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\overset{5/2}{-\!\!-\!\!-}\circ\cdots\circ$

This is not at all easy.....

Numerical computation

Use **Schläfli's volume differential formula** in the hyperbolic case :

$$d\,\mathrm{vol}_n = \frac{1}{1-n}\sum_F \mathrm{vol}_{n-2}(F)\,d\alpha_F$$

Together with Lobachevsky's formula, it allows to deduce the simple integral expression

$$\mathrm{vol}_5(\circ\overset{5}{\text{——}}\circ\text{——}\circ\text{——}\circ\text{——}\circ\text{——}\circ\cdots\circ) =$$

$$= \frac{1}{16}\int_{\pi/5}^{\alpha_0}\Big\{\,Л_2(\beta(t)+\theta(t)) - Л_2(\beta(t)-\theta(t)) +$$

$$+Л_2(\tfrac{\pi}{6}-\theta(t)) - Л_2(\tfrac{\pi}{6}+\theta(t)) +$$

$$+Л_2(\tfrac{\pi}{3}+\theta(t)) - Л_2(\tfrac{\pi}{3}-\theta(t)) + 2\,Л_2(\tfrac{\pi}{2}-\theta(t))\Big\}\,dt$$

$$\simeq 0.00076735958431780932334590186 21 \quad ,$$

where

$$\alpha_0 = \tfrac{1}{2}\arccos\tfrac{1}{5}$$

$$\beta(x) = \arccos\frac{\sin x}{\sqrt{4\sin^2 x - 1}}$$

$$\theta(x) = \arctan\sqrt{1-2\tan^2\beta(x)}\,.$$

The index two subgroup Γ_0 of orientation preserving isometries of $\circ\overset{5}{\text{——}}\circ\text{——}\circ\text{——}\circ\text{——}\circ\text{——}\circ\cdots\circ$ has also the **numerical** covolume

$$(\star) \qquad \cong 0.0015347191686356186466918037 24$$

The three smallest compact arithmetic 5-orbifolds

(III) By the same method (based on Prasad's volume formula), Emery computed the second and third smallest volumes of oriented compact **arithmetic** 5-orbifolds

Group	Hyperbolic covolume
Γ_0	$0.00153459236\ldots$ $(\star)$
Γ_1	$0.00306918472\ldots$
Γ_2	$0.00396939286\ldots$

Consequences :

(a) The orientation preserving subgroup Γ_0 in the Coxeter group

$\Gamma_* \, : \, \circ \overset{5}{\text{——}} \circ \text{——} \circ \text{——} \circ \text{——} \circ \text{——} \circ \cdots \circ$ has minimal covolume $\square$

(b) By **arithmetic** considerations : $[\Gamma_0 : \Gamma_1] = 2$

Geometrically : A fundamental polytope for the reflection subgroup in Γ_1 arises by doubling along the Coxeter face simplex with graph $\circ \overset{5}{\text{——}} \circ \text{——} \circ \text{——} \circ \text{——} \circ$ which is orthogonal to all neighboring facets

(c) Similarly to (a), for the arithmetic volume

$$\mathrm{vol}_3(\mathbb{H}^3/\Gamma_2) = \frac{9\sqrt{5}^{15}}{2^3\pi^{15}}\zeta_{k_0}(2)\zeta_{k_0}(4)L_{\ell_2/k_0}(3) \cong 0.00396939286$$

where $\ell_2 = \mathbb{Q}(\sqrt{\omega}) \cong \mathbb{Q}[x]/(x^4 - x^2 - 1)$, $\omega = \frac{1+\sqrt{5}}{2}$:

the value equals (numerically) **twice** the volume of the truncated Coxeter 5-prism

$$\circ \overset{5}{\text{——}} \circ \text{——} \circ \text{——} \circ \text{——} \circ \overset{4}{\text{——}} \circ \cdots \circ$$

Arithmetic proof ingredients

(as used by Belolipetsky for $4 \leq n \equiv 0$ (2) and
by Emery for $5 \leq n \equiv 1$ (2))

- Prasad's covolume formula for principal arithmetic subgroup

$$\mu(\text{Spin}(n,1)/\Lambda) = \mathcal{D}_k^{\frac{2r^2-r}{2}} \left(\frac{\mathcal{D}_\ell}{\mathcal{D}_k^{[\ell:k]}} \right)^{\frac{1}{2}(2r-1)} C(r)^{[F:\mathbb{Q}]} \, \mathcal{E}(\mathcal{P}),$$

where ℓ is the field above, $\mathcal{D}_K$ denotes the absolute value of the discriminant of a number field K, $\mathcal{E}(\mathcal{P}) = \prod_{v \in V_f} e_v({}_v)$ is an Euler product of certain local factors of Λ, and

$$C(r) = \frac{(r-1)!}{(2\pi)^r} \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}}.$$

- Work of Borel-Prasad
- Bruhat-Tits theory for the local factors in Prasad's formula
- some particular technical aspects and difficulties for **odd** n :

e.g., the algebraic group whose real points is $\text{PO}_0(n,1)$ has an algebraic simply connected covering being a 4-covering; the group $\text{PO}_0(n,1)$ is of type D, for which there exist outer forms.