Discrete groups of automorphisms of surfaces with given genus

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Let $\mathcal{S}_g$ be an orientable surface of genus g, (compact, connected,...)

Discrete group G acting on $\mathcal{S}$

- a group of self-homeomorphisms of $\mathcal{S}$, s.t. each orbit forms a discrete set
- a point stabiliser under an action of G is cyclic (dihedral $\Rightarrow$ non-orientable)
- compact connected surface $\mathcal{S}_g \iff G$ is finite,

Regular cover of a surface is $\mathcal{S}_g/G = \mathcal{S}_\gamma$ is a *quotient orbifold*

Quotient orbifold – an orientable surface of genus γ with r points distinguished, every branch-point is endowed with branch-index $m_i > 1$. Orbifold *signature* $(\gamma; \{m_1, \ldots, m_r\})$.

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Action of a discrete group

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How to relate a surface and its cover?

Euler characteristic of $\mathcal{S}_g$ is $\chi = 2 - 2g$;

Regular cover of a surface is $\mathcal{S}_g/G = \mathcal{S}_\gamma$, $\gamma \leq g$, a quotient orbifold

Smooth cover of an orientable surface ($|G|$-folded cover)

$$(2 - 2g) = |G|(2 - 2\gamma);$$

Quotient orbifold is an 'ordinary' surface of genus γ – no branch-points

Branched cover of an orientable surface (Riemann-Hurwitz equation)

$$2 - 2g = |G|\left(2 - 2\gamma - \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right);\ \forall i : m_i \geq 2 \in \mathbb{Z};\ m_i \,|\, |G|;$$

m_i is the order of the cyclic stabiliser of ith branch-point

Hurwitz condition: $|G| \leq 84(g - 1)$, when $\mathcal{S}$ is of genus $g > 1$, compact, connected, orientable

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Theorem (Koebe Theorem)

Every discrete group acting on $\mathcal{S}_g$ is an epimorphic image of $\pi_1(\mathcal{O})$ for some g-admissible orbifold $\mathcal{O}$ with signature $(\gamma; \{m_1, m_2, \ldots, m_r\})$.

$$\begin{tikzcd}
(\widetilde{S}, \pi_1(\mathcal{O})) \arrow[dr, "K"] \arrow[dd, "\mathrm{U}"'] & \\
& (S_g, G) \arrow[dl, "G"] \\
(\mathcal{O}(\gamma; \{m_1, \ldots, m_r\}), 1) &
\end{tikzcd}$$

The orbifold fundamental group $\pi_1(\mathcal{O})$ is a Fuchsian group F with signature $(\gamma; \{m_1, m_2, \ldots, m_r\})$

$$F = \langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma \mid x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = 1, \prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^{r} x_j = 1 \rangle.$$

Classification: the main idea

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Requirements: $g > 1$ - genus of a surface

Output: list of all actions of finite groups on $\mathcal{S}_g$.

1 solve Riemann-Hurwitz equation numerically;

2 construct Fuchsian groups $F(\gamma, \{m_1, \ldots, m_r\})$ given by solutions of (1);

low-index subgroups approach

3' search for all low-index normal subgroups of index $|G|$;

4' for every $K \trianglelefteq F$ test whether $\varepsilon : F \to F/K$ is order-preserving on elliptic generators of F; STOP.

or examining epimorphisms $F \to G$

3" given F and G construct all epimorphisms $F \to G$;

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Rieman-Hurwitz equation

$$2-2g=|G|\left(2-2\gamma-\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)$$

Criteria for a solution:

1. $\gamma \leq g$,
2. $r \leq 2g+2$,
3. $\forall i : m_i \geq 2 \in \mathbb{Z}$,
4. $\forall i : |G| \equiv 0 \bmod m_i$,
5. $|G| \leq 84(g-1)$.

We obtain a set of pairs (signature of an orbifold, order of respective group)

$$|G|, (\gamma; \{m_1, \dots, m_r\}).$$

Not every signature is g-admissible: RHe holds, but an action of G does not exist.

- $(0; \{7,3,2\})$ is not 2-admissible,
- $(0; \{5,4,2\})$ is not 3-admissible, etc...

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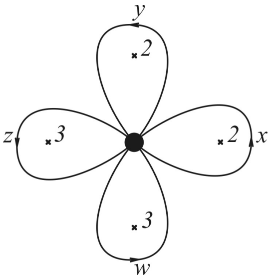
Arithmetics vs. group theory: genus 2 actions

$\lvert G\rvert$	Orbifold	Actions	$\lvert G\rvert$	Orbifold	Actions
1	$(2;\{\})$	1	12	$(0;\{3,2,2,2\})$	D_{12}
2	$(1;\{2,2\})$	C_2	12	$(0;\{4,4,3\})$	$C_3 : C_4$
2	$(0;\{2,2,2,2,2,2\})$	C_2	12	$(0;\{6,3,3\})$	—
3	$(1;\{3\})$	—	12	$(0;\{6,6,2\})$	$C_6 \times C_2$
3	$(0;\{3,3,3,3\})$	C_3	12	$(0;\{12,4,2\})$	—
4	$(1;\{2\})$	—	15	$(0;\{5,3,3\})$	—
4	$(0;\{2,2,2,2,2\})$	$C_2 \times C_2$	16	$(0;\{8,4,2\})$	QD_{16}
4	$(0;\{4,4,2,2\})$	C_4	18	$(0;\{18,3,2\})$	—
5	$(0;\{5,5,5\})$	C_5	20	$(0;\{5,5,2\})$	—
6	$(0;\{3,3,2,2\})$	$C_6,\ S_3$	24	$(0;\{4,3,3\})$	$SL(2,3)$
6	$(0;\{6,2,2,2\})$	—	24	$(0;\{6,4,2\})$	$(C_6 \times C_2) : C_2$
6	$(0;\{6,6,3\})$	C_6	24	$(0;\{12,3,2\})$	—
8	$(0;\{4,2,2,2\})$	D_8	30	$(0;\{10,3,2\})$	—
8	$(0;\{4,4,4\})$	Q_8	36	$(0;\{9,3,2\})$	—
8	$(0;\{8,8,2\})$	C_8	40	$(0;\{5,4,2\})$	—
9	$(0;\{9,3,3\})$	—	48	$(0;\{8,3,2\})$	$GL(2,3)$
10	$(0;\{10,5,2\})$	C_{10}	84	$(0;\{7,3,2\})$	—

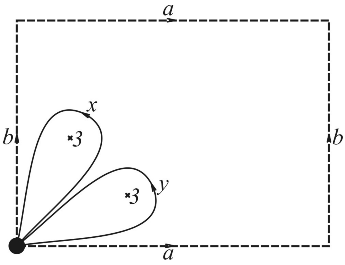
1. canonical quotient map $\bar{\mathbf{M}}$ is a bouquet of r loops on the surface of genus γ,
2. every loop is the boundary of a face containing exactly one branch-point with respective branch-index m_i,
3. outer face of the map is an $(r+2\gamma)$-gon; contains no branch-point.

$\mathcal{O}(0; \{2,2,3,3\})$



$\langle x, y, z, w \mid x^2 = y^2 = z^3 = w^3 = xyzw = 1\rangle$

$\mathcal{O}(1; \{3,3\})$



$\langle x, y, a, b \mid x^3 = y^3 = [a, b]xy = 1\rangle$

Main result

Previous results

Broughton '89 classification for genera 2 and 3 (not all actions shown)

Bogopolski '91 classification for genus 4

Kuribayashi and Kimura 90's classification for genus 5

State-of-art

- Abstract structure of groups and g-admissible orbifold types were dermined up to genus 24, for large groups much further (see Conder's web page, $|G| \geq 4(g-1)$)
- At present we have completed the list of actions up to genus 8, see `http://www.savbb.sk/~karabas/science.html#rhsu`
- Small 9-admissible troublemakers with more than 10^5 kernels: $C_2 \times C_2$ of types $(1; \{2^8\})$ or $(0; \{2^{12}\})$, $C_2 \times C_2 \times C_2$ of types $(1; \{4^2, 2\})$ or $(0; \{2^8\})$, D_8 of type $(0; \{2^8\})$

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- Small 9-admissible troublemakers with more than 10^5 kernels: $C_2 \times C_2$ of types $(1; \{2^8\})$ or $(0; \{2^{12}\})$, $C_2 \times C_2 \times C_2$ of types $(1; \{4^2, 2\})$ or $(0; \{2^8\})$, D_8 of type $(0; \{2^8\})$

Previous results

Broughton '89 classification for genera 2 and 3 (not all actions shown)

Bogopolski '91 classification for genus 4

Kuribayashi and Kimura 90's classification for genus 5

State-of-art

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Main result

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A sample of results: Small genera

Discrete groups with actions on $\mathcal{S}_0$

1	1	$\mathcal{O}(0;\emptyset)$	A_4	1	$\mathcal{O}(0;\{2,3,3\})$
C_m	$\mathrm{Epi}_O(\pi_1(\mathcal{O}), C_m)$	$\mathcal{O}(0;\{m,m\})$	S_4	1	$\mathcal{O}(0;\{2,3,4\})$
D_{2m}	?	$\mathcal{O}(0;\{2,2,m\})$	A_5	1	$\mathcal{O}(0;\{2,3,5\})$

By R.-H. bound, for $g > 1$ there are finitely many g-admissible orbifolds.
see `http://www.savbb.sk/~karabas/science.html#rhsu`, for $2 \leq g \leq 24$

Discrete groups with actions on $\mathcal{S}_2$

1	1	$(2;\emptyset)$	Q_8	6	$(0;\{4,4,4\})$
C_2	1	$(0;\{2^6\})$	D_8	6	$(0;\{2^3,4\})$
C_2	4	$(1;\{2,2\})$	C_{10}	4	$(0;\{2,5,10\})$
C_3	6	$(0;\{3^4\})$	$C_2 \times C_6$	12	$(0;\{2,6,6\})$
C_4	2	$(0;\{2^2,4^2\})$	$C_3 \rtimes C_4$	2	$(0;\{3,4,4\})$
$C_2 \times C_2$	60	$(0;\{2^5\})$	D_{12}	6	$(0;\{2^3,3\})$
C_5	12	$(0;\{5,5,5\})$	$C_8 \rtimes C_2$	2	$(0;\{2,4,8\})$
C_6	2	$(0;\{3,6,6\})$	$C_2 \ltimes (C_2 \times C_6)$	2	$(0;\{2,4,6\})$
C_6	2	$(0;\{2^2,3^2\})$	$\mathrm{SL}_2(3)$	2	$(0;\{3,3,4\})$
S_3	2	$(0;\{2^2,3^2\})$	$\mathrm{GL}_2(3)$	2	$(0;\{2,3,8\})$
C_8	4	$(0;\{2,8,8\})$			

A sample of results: Numbers of admissible pairs (group-signature)

g	# adm. pairs	max. $\lvert \mathrm{G} \rvert$	g	# adm. pairs	max. $\lvert \mathrm{G} \rvert$
2	21	48	14	229	1092
3	49	168	15	407	504
4	64	120	16	386	720
5	93	192	17	$>$732	1344
6	87	150	18	337	168
7	148	504	19	789	720
8	108	336	20	425	228
9	270	320	21	940	480
10	226	432	22	628	1008
11	232	240	23	716	192
12	201	120	24	625	216
13	454	360			

g	$\lvert G \rvert$	Orbifold	$\mathrm{Epi}_{\mathcal{O}}(\mathcal{S}_g, \mathrm{G})$	G
2	48	$(0, \{8, 3, 2\})$	2	$GL(2, 3)$
3	168	$(0, \{7, 3, 2\})$	2	$PSL(3, 2)$
4	120	$(0, \{5, 4, 2\})$	1	S_5
5	192	$(0, \{8, 3, 2\})$	4	$(((C_4 \times C_2) : C_4) : C_3) : C_2$
6	150	$(0, \{10, 3, 2\})$	4	$((C_5 \times C_5) : C_3) : C_2$
7	504	$(0, \{7, 3, 2\})$	3	$PSL(2, 8)$
8	336	$(0, \{8, 3, 2\})$	2	$PSL(3, 2) : C_2$
9	320	$(0, \{5, 4, 2\})$	4	$(((C_2 \times Q_8) : C_2) : C_5) : C_2$
10	432	$(0, \{8, 3, 2\})$	2	$(((C_3 \times C_3) : Q_8) : C_3) : C_2$
11	240	$(0, \{6, 4, 2\})$	2	$C_2 \times S_5$
12	120	$(0, \{15, 4, 2\})$	4	$(C_5 \times A_4) : C_2$
13	360	$(0, \{10, 3, 2\})$	2	$A_5 \times S_3$
14	1092	$(0, \{7, 3, 2\})$	6	$PSL(2, 13)$
15	504	$(0, \{9, 3, 2\})$	3	$PSL(2, 8)$

g	$\lvert\mathrm{G}\rvert$	Orbifold	$\mathrm{Epi}_{\mathcal{O}}(\mathcal{S}_g, \mathrm{G})$	G
16	720	$(0, \{8, 3, 2\})$	2	$A_6 : C_2$
17	1344	$(0, \{7, 3, 2\})$	2	$(C_2 \times C_2 \times C_2).PSL(3, 2)$
18	168	$(0, \{21, 4, 2\})$	6	$(C_7 \times A_4) : C_2$
19	720	$(0, \{5, 4, 2\})$	4	$C_2 \times A_6$
20	228	$(0, \{6, 6, 2\})$	24	$C_2 \times ((C_{19} : C_3) : C_2)$
21	480	$(0, \{6, 4, 2\})$	2	$(C_2 \times C_2 \times A_5) : C_2$
22	1008	$(0, \{8, 3, 2\})$	4	$(C_3 \times PSL(3, 2)) : C_2$
23	192	$(0, \{48, 4, 2\})$	8	$(C_3 \times (C_{16} : C_2)) : C_2$
24	216	$(0, \{27, 4, 2\})$	9	$((C_2 \times C_2) : C_{27}) : C_2$

All shown maximal actions are 'triangular'

Question: Does there exist a maximal action possessing non-triangular g-admissible signature?

YES: $g = 126$, $|G| = 1500$, $(0; \{3, 2^3\})$ (Conder)

g	$\lvert \mathrm{G} \rvert$	Orbifold	$\mathrm{Epi}_{\mathcal{O}}(\mathcal{S}_g, \mathrm{G})$	G
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g	Signature	$\#C_2$	$\#C_4 \times C_2$	g	Signature	$\#C_2$	$\#C_4 \times C_2$
2	$(1, \{2^2\})$	4		6	$(2, \{2^6\})$	16	
2	$(0, \{2^6\})$	1		6	$(1, \{2^{10}\})$	4	
3	$(2, \{\})$	15		6	$(0, \{2^{14}\})$	1	
3	$(1, \{2^4\})$	4		7	$(4, \{\})$	255	
3	$(0, \{2^8\})$	1		7	$(3, \{2^4\})$	64	
3	$(0, \{4^2, 2^2\})$		32	7	$(2, \{2^8\})$	16	
4	$(2, \{2^2\})$	16		7	$(1, \{2^{12}\})$	4	
4	$(1, \{2^6\})$	4		7	$(0, \{2^{16}\})$	1	
4	$(0, \{2^{10}\})$	1		7	$(1, \{2^3\})$		288
5	$(3, \{\})$	63		7	$(1, \{4^2\})$		192
5	$(2, \{2^4\})$	16		7	$(0, \{4^2, 2^4\})$		320
5	$(1, \{2^8\})$	4		7	$(0, \{4^4, 2\})$		176
5	$(0, \{2^{12}\})$	1		8	$(4, \{2^2\})$	256	
5	$(1, \{2^2\})$		120	8	$(3, \{2^6\})$	64	
5	$(0, \{4^2, 2^3\})$		104	8	$(2, \{2^{10}\})$	16	
5	$(0, \{4^4\})$		48	8	$(1, \{2^{14}\})$	4	
6	$(3, \{2^2\})$	64		8	$(0, \{2^{18}\})$	1	