On the Disconnectedness of the Branch Loci of Moduli Spaces of Riemann Surfaces

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(X, complex atlas) $X \equiv \frac{\mathcal{H}}{\Delta}$, with Δ a (cocompact) Fuchsian group
Surface Fuchsian Group $\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \,|\, \Pi[a_i, b_i] = 1 \rangle$

Teichmüller space $\mathcal{T}_g$, space of geometries on a surface of genus g
$\mathcal{T}_g = \{\sigma : \Gamma_g \to PSL(2, \mathbb{R}) \,|\, \sigma \, injective, \, \sigma(\Gamma_g) \text{ discrete } \}/PSL(2, \mathbb{R})$

Moduli space $\mathcal{M}_g$, space (orbifold) of conformal structures on a surface of genus g

Mapping Class Group (Teichmüller Modular Group)
$M_g = \frac{Diff(X)}{Diff_0(X)} = Out(\Gamma_g)$

Orbifold Universal Covering $\mathcal{M}_g = \mathcal{T}_g / M_g$
$\mathcal{B}_g$ Branching Locus = Singular Locus of $\mathcal{M}_g$

$\mathcal{B}_g = \{X \in \mathcal{M}_g \,|\, Aut(X) \neq 1\}$

$g = 1$ Euclidean case: $\mathcal{T}_1 = \mathcal{H}, M_1 = PSL(2, \mathbb{Z}), \mathcal{B}_1 = \{i, e^{i\pi/3}\}$,

Results: The branch loci $\mathcal{B}_g$ of moduli spaces of hyperbolic Riemann surfaces are disconnected for all genera with the exception of genera **3**, **4**, **7**, **13**, **17**, **19** and **59**.

For genera greater or equal than sixty the biggest (Teichmüller-) dimension of an isolated stratum is:
$\frac{g-2}{2}$ if g even, formed by pentagonal surfaces ($g \geq 18$);
$\frac{g-1}{2}$ if $g \equiv 1 \bmod 4$, formed by elliptic-pentagonal surfaces ($g \geq 29$, $g \neq 37$);
$\frac{g-3}{3}$ if $g \equiv 0 \bmod 3$, $3 \bmod 4$; formed by heptagonal surfaces ($g \geq 39$);
$\frac{g-1}{3}$ if $g \equiv 1 \bmod 3$, $3 \bmod 4$; formed by elliptic-heptagonal surfaces ($g \geq 52$);
$\frac{g+1}{3}$ if $g \equiv 2 \bmod 3$, $3 \bmod 4$; formed by 2-elliptic-heptagonal surfaces ($g \geq 65$).

Fuchsian Groups

Δ (cocompact) discrete subgroup of $PSL(2,\mathbb{R})$
A (compact) Riemann Surface of genus $g \geq 2$ $\quad X = \frac{\mathcal{H}}{\Delta}$
Δ has presentation:
generators: $x_1, ..., x_r, a_1, b_1, ..., a_h, b_h$
relations: $x_i^{m_i}, i = 1 : r,\ x_1...x_r a_1 b_1 a_1^{-1} b_1^{-1} ... a_h b_h a_h^{-1} b_h^{-1}$
$X = \frac{\mathcal{H}}{\Delta}$: orbifold with r cone points and underlying surface of genus g
Algebraic structure of Δ and geometric structure of X are determined by the signature $\quad s(\Delta) = (h; m_1, \ldots, m_r)$
Area of Δ: area of a fundamental region P
$\mu(\Delta) = 2\pi(2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$
X hyperbolic equivalent to $P/\langle\text{pairing}\rangle$

Topological equivalence

G finite group of automorphisms of $X_g = \mathcal{H}/\Gamma_g$, Γ_g a surface Fuchsian group iif there exist
Δ Fuchsian group and epimorphism $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma_g$
θ is the monodromy of the covering $f : \mathcal{H}/\Gamma_g \to \mathcal{H}/\Delta$

$$\begin{array}{ccc} & \mathcal{H} & \\ \swarrow & & \\ X/ = \mathcal{H}/\Gamma_g & \downarrow & \Delta\text{: lifting to } \mathcal{H} \text{ of } G \\ \searrow & & \\ & X/G = \mathcal{H}/\Delta & \end{array}$$

An automorphism of X_g will fix the class of the uniformizing Fuchsian group

A morphism $f : X = \mathcal{H}/\Lambda \to Y = \mathcal{H}/\Delta$, X, Y Riemann surfaces, group inclusion $i : \Lambda \to \Delta$
Covering f determined by monodromy $\theta : \Delta \to \Sigma_N$,
$\Lambda = \theta^{-1}(STb(1))$
(symbol $\leftrightarrow$ Λ-coset $\leftrightarrow$ sheet for f)

Theorem (Singerman 1971) Λ (and so i) determined θ (and Δ): If $s(\Delta) = (h; m_1, \dots, m_r)$, then
$s(\Lambda) = (h'; m'_{11}, \dots, m'_{1s_1}, \dots, m'_{r1}, \dots, m'_{rs_r})$ iff $\theta : \Delta \to \Sigma_{|\Delta:\Lambda|}$ s.t.
i) Riemann-Hurwitz $\frac{\mu(\Lambda)}{\mu(\Delta)} = |\Delta : \Lambda|$
ii) $\theta(x_i)$ product of s_i cycles each of length $\frac{m_i}{m'_{i1}}, \dots, \frac{m_i}{m'_{is_i}}$

p-gonal Riemann Surfaces

A Riemann surface X is called *p-gonal* if it admits a morphism of degree p on the Riemann sphere
X is called cyclic p-gonal when X has an automorphism φ of order p such that $X/\langle\varphi\rangle = \hat{\mathbb{C}}$.
Case $p = 2$: X hyperelliptic R.S.
A Riemann surface X is called *elliptic-p-gonal* if it admits a morphism of degree p on a torus.
X is called cyclic elliptic-p-gonal when the morphism is a regular covering.
Severi-Castelnuovo inequality: A p-gonal morphism of X is unique if the genus of $X \geq (p-1)^2$.
An elliptic-p-gonal morphism of X is unique if the genus of $X \geq 2p + (p-1)^2$.

Teichmüller and Moduli Spaces

Δ abstract Fuchsian group $\quad s(\Delta) = (h; m_1, \dots, m_r)$
$\mathcal{T}_\Delta = \{\sigma : \Delta \to PSL(2,\mathbb{R}) \mid \sigma\, injective,\ \sigma(\Gamma_g)$ discrete $\}/PSL(2,\mathbb{R})$
Teichmüller space $\mathcal{T}_\Delta$ has a complex structure of dim $3h-3+r$, diffeomorphic to a ball of dim $6h-6+2r$.

If Λ subgroup of Δ $(i : \Lambda \to \Delta) \Rightarrow i_* : \mathcal{T}_\Delta \to \mathcal{T}_\Lambda$ embedding
Γ_g surface Fuchsian group $\quad \Gamma_g \leq \Delta \quad \mathcal{T}_\Delta \subset \mathcal{T}_{\Gamma_g} = \mathcal{T}_g$
G finite group $\quad \mathcal{T}_g^G = \{[\sigma] \in \mathcal{T}_g \mid g[\sigma] = [\sigma]\, \forall g \in G\} \neq \emptyset$
$\mathcal{T}_g^G$: surfaces with G as a group of automorphisms.

Mapping class group $M(\Delta) = Out(\Delta) = \frac{Diff(\mathcal{H}/\Delta)}{Diff_0(\mathcal{H}/\Delta)}$
$\Delta = \pi_1(\mathcal{H}/\Delta)$ as orfibold
$M(\Delta)$ acts properly discontinuously on $\mathcal{T}_\Delta$
$\mathcal{M}_\Delta = \mathcal{T}_\Delta / M(\Delta)$

Surfaces with automorphisms : **Branch Locus**

Consider a marked surface $\sigma(X) \in \mathcal{T}_g$ and $\beta \in M_g$, we have
$\beta[\sigma] = [\sigma] \quad \Leftrightarrow \quad \gamma \in PSL(2\mathbb{R}), \quad \sigma(\Gamma_g) = \gamma^{-1}\sigma\beta(\Gamma_g)\gamma$
γ induces an automorphism of $[\sigma(X)]$

$Stb_{\mathcal{M}_g}[\sigma] = \{\beta \in M_g \mid \beta[\sigma] = [\sigma]\} = Aut([\sigma(X)])$
$G = Aut(X)$ finite, determines a conjugacy class of finite subgroups of M_g, the **symmetry** of X

X_g, Y_g equisymmetric if $Aut(X_g)$ conjugate to $Aut(Y_g)$
($Aut(X_g)$: **full automorphism group**.
Singerman's list of non-maximal signatures.

Action: $\theta : \Delta \to Aut(X_g) = G$, $ker(\theta) = \Gamma_g$
$Aut(X) = G$ conjugate $Aut(Y) \quad \Leftrightarrow \quad w \in Aut(G), h \in Diff(X_0)$
$\epsilon, \epsilon' : G \to Diff(X_0), \epsilon'(g) = h\epsilon w(g)h^{-1}$
Two (surface) monodromies $\theta_1, \theta_2 : \Delta \to G$ topologically equiv.

$$\begin{array}{ccccc} & \Delta & \stackrel{\theta_1}{\to} & G & \\ \text{actions of G} \quad \beta \in Aut(\Delta) & \downarrow & & \downarrow & w \in Aut(G) \\ & \Delta & \stackrel{\theta_2}{\to} & G & \end{array}$$

θ_1, θ_2 equiv under $\mathcal{B}(\Delta) \times Aut(G)$, $\mathcal{B}(\Delta)$ **braid group**

Broughton (1990): **Equisymmetric Stratification**
$\mathcal{M}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{is} G\}$
$\overline{\mathcal{M}}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{contains} G\}$
$\mathcal{M}_g^{G,\theta}$ smooth, connected, locally closed al. subvar. of $\mathcal{M}_g$, dense in $\overline{\mathcal{M}}_g^{G,\theta}$ $\qquad \mathcal{B}_g = \cup \overline{\mathcal{M}}_g^{G,\theta}$
Costa-I (2008) $\mathcal{B}_g = \cup \mathcal{M}_g^{C_p,\theta}$ (Cornalba 1987 and 2008)
We need to look at maximal actions of C_p

Connectedness, we are interested in $Y \in \overline{\mathcal{M}}_g^{G_1,\theta_1} \cap \overline{\mathcal{M}}_g^{G_2,\theta_2}$

Finding $\theta : \Delta \to G = Aut(Y)$ extends both $\theta_1 : \Delta_1 \to G_1$ and $\theta_2 : \Delta_2 \to G_2$ with $Ker(\theta) = Ker(\theta_1) = Ker(\theta_2) = \Gamma_g$

Geometrically

$$\begin{tikzcd} & Y = \mathcal{H}/\Gamma_g \arrow[dl, "f_{\theta_1}"'] \arrow[d] \arrow[dr, "f_{\theta_2}"] & \\ \mathcal{H}/\Delta_1 \arrow[dr] & f_\theta \arrow[d] & \mathcal{H}/\Delta_2 \arrow[dl] \\ & \mathcal{H}/\Delta & \end{tikzcd}$$

$G_1 = C_{p_1}$ and $G_2 = C_{p_2}$

Corresponding diagramme for embeddings of finite groups

Costa-I (2008) $\mathcal{B}_4$ is connected
Kulkarni (1991). Existence of isolated points in $\mathcal{B}_g$ iff $g = 2$ or 2g+1 a prime ≥ 11
Isolated points are given by actions
$\theta : \Delta(0; p, p, p) \to C_p,\ p = 2g + 1$
The actions of C_7 in $\mathcal{M}_3$ extend to actions of C_{14} or $PSL(2, 7)$

Bartolini-I (2009): $\overline{\mathcal{M}}_g^{C_2,\theta}$ and $\overline{\mathcal{M}}_g^{C_3,\theta'}$ belong to the same connected component of $\mathcal{B}_g$.
All the closed strata induced by actions of C_2 or C_3 intersect the closed stratum formed by surfaces X_g admitting an automorphism of order 2 with quotient Riemann surface of genus highest possible:
$\frac{g}{2}$ for even g,
$\frac{g+1}{2}$ for odd g.

Costa-I (2009): $\mathcal{B}_g$ contains isolated strata of dimension 1 iff g+1 is a prime ≥ 11

The isolated strata are given by actions:
$\theta_i : \Delta(0; p, p, p, p) \to C_p$ with $\theta_i(x_1) = a, \theta_i(x_2) = a^i, \theta_i(x_3) = a^j, \quad i \neq 1, p-1, j \neq 1, p-1, i, i-1.$

This case does not exist for $p = 5$ and $p = 7$

These actions are maximal and the strata contain no curve with more symmetry.

Branch loci in genera four, seven, thirteen, seventeen, nineteen and fiftynine are connected. GAP-machinery !!

Bartolini-Costa-I (2011) These are the only genera with connected branch locus.

Actions given isolated stratum of maximal dimension

$\mathbf{g} = \mathbf{60}$, action $\theta : \Delta(0; 5^{32}) \to C_5$:
$\theta(x_1) = \cdots = \theta(x_{19}) = \alpha$, $\theta(x_{20}) = \cdots = \theta(x_{24}) = \alpha^2$,
$\theta(x_{25}) = \alpha^3$, $\theta(x_{26}) = \cdots = \theta(x_{32}) = \alpha^4$.
$\mathbf{g} = \mathbf{61}$, action $\theta : \Delta(1; 5^{30}) \to C_5$
$\theta(a) = \theta(b) = 1$, $\theta(x_1) = \cdots = \theta(x_{23}) = \alpha$,
$\theta(x_{24}) = \cdots = \theta(x_{28}) = \alpha^2$, $\theta(x_{29}) = \alpha^3$, $\theta(x_{30}) = \alpha^4$.
$\mathbf{g} = \mathbf{63}$, action $\theta : \Delta(0; 7^{23}) \to C_7$:
$\theta(x_1) = \cdots = \theta(x_{14}) = \alpha$, $\theta(x_{15}) = \cdots = \theta(x_{19}) = \alpha^5$,
$\theta(x_{20}) = \alpha^4$, $\theta(x_{21}) = \cdots = \theta(x_{23}) = \alpha^2$.
$\mathbf{g} = \mathbf{67}$, action $\theta : \Delta(1; 7^{22}) \to C_7$
$\theta(a) = \theta(b) = 1$, $\theta(x_1) = \cdots = \theta(x_{17}) = \alpha$,
$\theta(x_{18}) = \cdots = \theta(x_{20}) = \alpha^6$, $\theta(x_{21}) = \alpha^3$, $\theta(x_{22}) = \alpha^4$.
$\mathbf{g} = \mathbf{71}$, action $\theta : \Delta(2; 7^{21}) \to C_7$
$\theta(a_i) = \theta(b_i) = 1, i = 1, 2$, $\theta(x_1) = \cdots = \theta(x_{13}) = \alpha$,
$\theta(x_{14}) = \cdots = \theta(x_{16}) = \alpha^2$, $\theta(x_{17}) = \theta(x_{18}) = \alpha^5$, $\theta(x_{19}) = \alpha^3$,
$\theta(x_{20}) = \alpha^4$, $\theta(x_{21}) = \alpha^6$.

THANK YOU