Distinguishing Infinite Graphs

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Most of the cited results about infinite graphs are joint work with

Janja Jerebic, Sandi Klavžar, Tom Tucker, Vladimir Trofimov, and Mark Watkins,

and the newest ones have been obtained together with

Johannes Cuno and Florian Lehner

Contents

1. Definitions and selected results for finite graphs

The distinguishing number $D(G)$ of a graph G
is the least cardinal number $\aleph$
such that G has a labeling with $\aleph$ labels
that is preserved only by id of Aut(G).

Definition by Albertson and Collins 1996

$D(G) \le \Delta + 1$, equality holding iff G is a K_n, $K_{n,n}$ or C_5

The definition also works for groups A acting on a set V.

We often say colors instead of labels. If a labeling ℓ is not preserved by an $\alpha \in A$, we say that ℓ breaks α.

Lemma *Let G be finite graph. Then, for every $A \subseteq \mathrm{Aut}(G) \setminus \{1\}$, there exists a two-coloring of the vertices of G that breaks at least half of the elements of A.*

The motion $m(\alpha)$ of a nontrivial permutation α, resp. automorphism α of a graph G, is the number of elements it moves.

$$m(\alpha) = |\{v \in V(G) : \alpha(v) \neq v\}|$$

The motion of a graph G is

$$m(G) = \min_{\alpha \in Aut(G) \setminus id} m(\alpha)$$

For example, $m(C_4) = 2, m(C_5) = 4, m(C_{100}) = 98, m(K_{100}) = 2$.

Theorem [Russell and Sundaram 1998]
If $m(G) > 2\log_2 |Aut(G)|$, then G is two-distinguishable.

The proof is very short, elegant and probabilistic. We will use this result, but prefer it in the form: If

$$2^{\frac{m(G)}{2}} > |\mathrm{Aut}(G)|\,, \quad \text{then} \quad D(G) = 2\,.$$

For example, $D(C_{10}) = 2$ because $2^{\frac{8}{2}} > 15 = |\mathrm{Aut} C_{10}|$.

A stronger form is

$$d^{\frac{m(G)}{2}} > |\mathrm{Aut}(G)| \Longrightarrow D(G) = d\,.$$

Actually one can show

$$d^{\frac{m(G)}{2}} \geq |\mathrm{Aut}(G) \setminus \{\mathrm{id}\}| \Longrightarrow D(G) = d.$$

Replacement of $\mathrm{Aut}(G)$ by $\mathrm{Aut}(G) \setminus \{\mathrm{id}\}$
clear by the arguments of Russell and Sundaram,

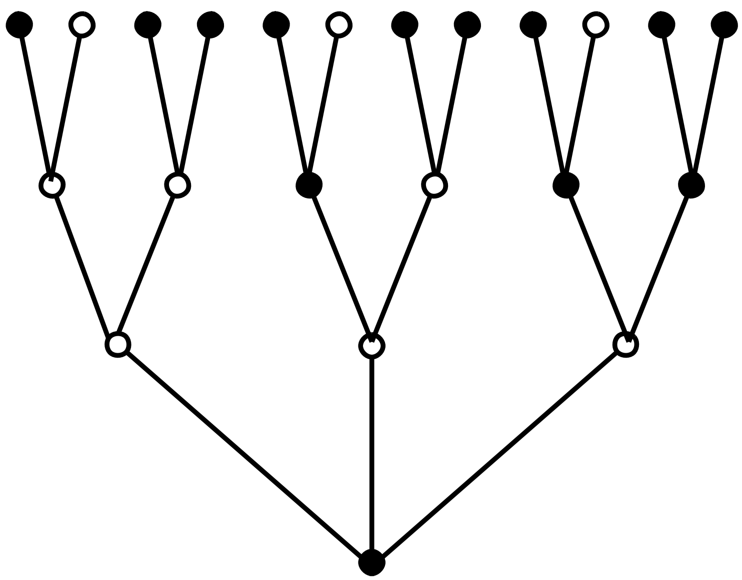
Further replacement of $>$ by $\geq$ requires some interesting argument.

Example: $D(C_5) = 3$
because $m(C_5) = 4$, $|\mathrm{Aut}(C_5)| = 10$ and $3^{4/2} = 9 = 10 - 1$.

Important for us:

large $m(G)$ implies small $D(G)$.

2. Infinite graphs



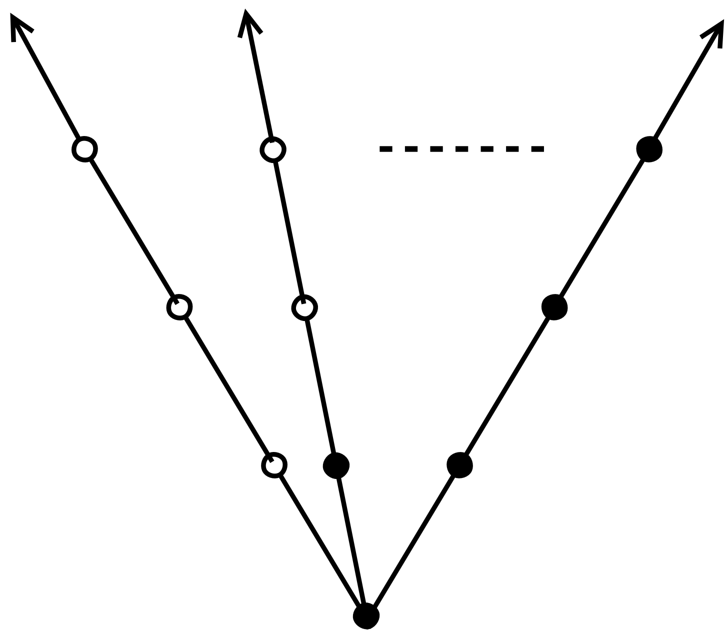
Consider the infinite homogeneous tree T_3

$D(T_3) = 2$

$|\mathrm{Aut}(T_3)| = \mathfrak{c}$

$m(T_3) = \aleph_0$

The result $D(T_3) = 2$ is already due to Babai 1977



Consider the star $S^*_{\aleph_0}$ consisting of $\aleph_0$ rays.

color by the 0,1 sequences

$D(S^*_{\aleph_0}) = 2$

$m(S^*_{\aleph_0}) = \aleph_0$

Notice that $D(S^*_{\mathfrak{n}}) \geq \aleph_0$ if $\mathfrak{n} > \mathfrak{c}$,

but $m(S^*_{\mathfrak{n}}) = \aleph_0$

3. The Infinite Motion Conjecture

Infinite Motion Conjecture [Tom Tucker] Let G be an infinite, locally finite, connected graph. If G has infinite motion, then $D(G) = 2$.

Theorem *The infinite motion conjecture is true if Aut(G) is countable.*

Proof Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\} = \text{Aut}(G)$.
Color all vertices white.

Induction with respect to n

Let $n = 1$.

α_1 has infinite motion, $\exists v_1 \in V(G). \ni .v_1 \neq \alpha_1 v_1$.

Color v_1 black. This breaks α_1.

Let $n \geq 1$. Suppose

$B = \{v_1, v_2, \ldots v_n\}$ are black,
$B' = \{\alpha_1 v_1, \alpha_2 v_2, \ldots \alpha_n v_n\}$ white,
$|B \cup B'| = 2n$.

Since $m(\alpha_{n+1}) = \infty$ there are infinitely many v such that $v \neq \alpha_{n+1} v$.

Thus $\exists v_{n+1}$ such that
$v_{n+1} \neq \alpha_{n+1} v_{n+1}$ and
$v_{n+1}, \alpha_{n+1} v_{n+1} \notin B \cup B'$. Color v_{n+1} black. $\square$

How good is this? It does not even cover T_3, nor S_∞^*.

Corollary *Let G be a locally finite, infinite, 3-connected planar graph. Then* $D(G) = 2$

Proof Such graphs have infinite motion and countable group. $\square$

We look for other classes of graphs with uncountable group for which the Infinite Motion Conjecture holds.

We might look for graphs that are close to finite ones, say graphs of polynomial growth.

4. Graph growth

Let G be locally finite, infinite, connected; $v_0 \in V(G)$
$B_n(v_0) = \{v \in V(G) : d(v_0, v) \le n\}$.

If $\exists k$ such that $|B_n(v_0)| = O(n^k)$, then G has polynomial growth k.

Definition independent of the choice of v_0.
Clear what we mean by linear and quadratic growth.

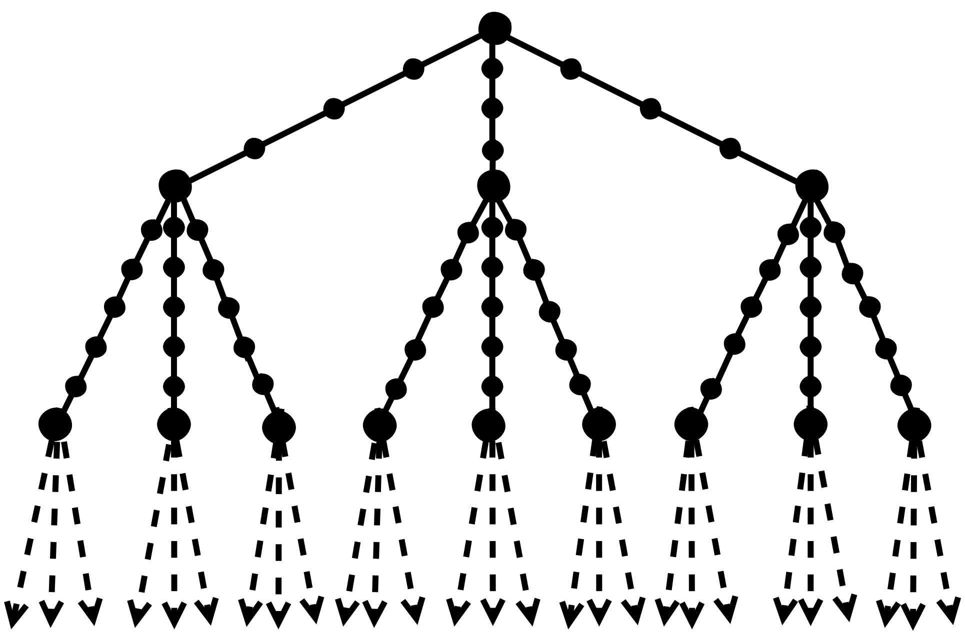
Linear growth means that $S_n(v_0) = \{v \in V(G) : d(v_0, v) = n\}$
is constant (on the average).

Theorem *Graphs of linear growth with infinite motion have countable group, and thus distinguishing number 2.*

5. Superlinear growth

Let $f(n)$ be a function that grows faster than any linear function. G has growth $f(n)$ if

$$|B_n(v_0)| = O(f(n))$$



There are graphs of such growth for any $f(n)$ that have uncountable group.

Just replace the edges of T_3 by paths of increasing lengths.

Lemma *For any $f(n)$ of superlinear growth there is a graph of growth $f(n)$, infinite motion and uncountable group.*

Proposition *Let G be a locally finite, infinite, connected graph with infinite motion. If there is $v \in V(G)$ such that*

$$|S_n(v)| \leq \frac{n}{4\log_2 n},$$

then $D(G) = 2$.

Two proofs, one with the lemma that there are always two-colorings that break at least half of the automorphisms, the other one with the Theorem of Russell and Sundaram.

Theorem *Let G be a locally finite, infinite, connected graph with infinite motion of growth $o(n^2/8\log_2 n)$, then $D(G) = 2$.*

Notice that the group of G can be uncountable.

6. Graphs with unbounded degrees and uncountable graphs

$D(R) = 2$, where R is the countable random graph. Tricky proof.

$D(Q_{\mathfrak{n}}) = 2$, where $Q_{\mathfrak{n}}$ is the hypercube of dimension $\mathfrak{n}$. Transfinite induction.

$D(K_\infty \Box K_\infty) = 2$, where $K_\infty \Box K_\infty$ is the Cartesian product of two countable complete graphs. Very easy proof.

$D(K_{\mathfrak{n}} \Box K_{\mathfrak{n}}) = 2$, proof by transfinite induction.

$D(K_{\mathfrak{n}} \Box K_{2^{\mathfrak{n}}}) = 2$, transfinite induction.

$D(K_{\mathfrak{n}} \Box K_{\mathfrak{m}}) > \mathfrak{n}$ if $\mathfrak{m} > 2^{\mathfrak{n}}$. Notice that $m(K_{\mathfrak{n}} \Box K_{\mathfrak{m}}) = \mathfrak{n}$.

7. The Infinite Motion Conjecture for uncountable graphs

How should we generalize

$$2^{\frac{m(G)}{2}} \geq |\mathrm{Aut}(G) \setminus \{\mathrm{id}\}| \Longrightarrow D(G) = 2.$$

to infinite graphs? Clearly $2^{m(G)} \geq |\mathrm{Aut}(G)| \Longrightarrow D(G) = 2.$

Infinite Motion Conjecture for uncountable graphs Let G be a connected uncountable graph. Then $2^{m(G)} \geq |\mathrm{Aut}(G)|$ implies $D(G) = 2.$

Tom Tucker's conjecture is $m(G) = \aleph_0 \Longrightarrow D(G) = 2$, where G is locally finite, infinite and connected.

Clearly G is countable, and thus $\mathrm{Aut}(G) \leq \mathfrak{c}$. If we assume the generalized continuum hypothesis, then $2^{\aleph_0} = \mathfrak{c}$, and thus $m(G) = \aleph_0$ implies $2^{\aleph_0} \geq |\mathrm{Aut}(G)|$.

8. A provable case for uncountable graphs

Infinite Motion Conjecture [Tom Tucker] Let G be a locally finite, infinite connected graph. If $2^{m(G)} \geq |\mathrm{Aut}(G)|$, then $D(G) = 2$.

We could prove that the conjecture is true for $m(G) = \aleph_0$ and $2^{m(G)} > |\mathrm{Aut}(G)|$.

Theorem *Let G be a connected, infinite graph. If $2^{m(G)} > |Aut(G)|$, then $D(G) = 2$.*

Proof By transfinite induction.