On finite groups of self-homeomorphisms of compact topological surfaces with invariant subsets

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Introduction

A finite group G of self-homeomorphisms of a closed surface X - orientable or not - of topological genus $g \geq 2$ has at most 84(g-1) elements while if such surface has $k \neq 0$ boundary components then it has at most $12(p-1)$ elements for so called algebraic genus $p = \varepsilon g + k - 1$ being bigger than 1 (where $\varepsilon = 2$ or 1 according to the surface being orientable or not).

If such action allow some essential invariant subsets (essential $\equiv$ their sizes smaller than $|G|$) then these bounds can be essential ly improved and the aim of this paper is to present known results concerning this subject.

Presented results can be proven by Nielsen-Riemann approach which allow to to see such groups as groups of automorphisms of such surfaces with some conformal structures imposed on them and use algebraic methods due to he Riemann uniformization theorem and well developed theory of of discrete groups of isometries of the hyperbolic plane.

Nielsen

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Given a finite group of self homeomorphisms of a compact topological surface X there exists a structure of a Riemann surface on X so that G is a group of conformal automorphisms of X.

Riemann

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$X = X_g$

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$$\begin{array}{c} \widetilde{X} \\ p \downarrow \\ X = X_g \end{array}$$

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$\Downarrow$ Riemann mapping theorem

$\widetilde{X} = \mathbb{C} \cup \{\infty\} \quad X = \mathbb{C} \quad X = \mathbb{C}^+ = \mathcal{H}$

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$$\underset{g=0}{\widetilde{X} = \mathbb{C} \cup \{\infty\}} \quad \underset{g=1}{X = \mathbb{C}} \quad \underset{g>1}{X = \mathbb{C}^+ = \mathcal{H}}$$

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$\widetilde{X}$ simply connected surface Riemann

$\Downarrow$ Riemann mapping theorem

$$\begin{array}{ccc} \widetilde{X} = \mathbb{C} \cup \{\infty\} & X = \mathbb{C} & X = \mathbb{C}^+ = \mathcal{H} \\ g = 0 & g = 1 & g > 1 \end{array}$$

$$\begin{array}{c} \Gamma \; \mathcal{H} \; \Delta \\ p \downarrow \\ X \;\; G \end{array}$$

Riemann

$$
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\end{array}
\qquad
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\Downarrow \text{Riemann mapping theorem} \\
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\end{array}
$$

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$$

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$\Gamma \leq \mathrm{Aut}(\mathcal{H}),\ \mathcal{H}/\Gamma \cong X,$

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$$\mathrm{Aut}(\mathcal{H}) = \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathcal{H})$$

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$$\begin{array}{ll} \widetilde{X} & \widetilde{X} \text{ simply connected surface Riemann} \\ p\downarrow & \qquad\qquad \Downarrow \text{Riemann mapping theorem} \\ X = X_g & \underset{g=0}{\widetilde{X} = \mathbb{C}\cup\{\infty\}} \quad \underset{g=1}{X = \mathbb{C}} \quad \underset{g>1}{X = \mathbb{C}^+ = \mathcal{H}} \end{array}$$

$$\begin{array}{ll} \Gamma\ \mathcal{H}\ \Delta & \Gamma \leq \mathrm{Aut}(\mathcal{H}),\ \mathcal{H}/\Gamma \cong X,\ \Gamma \text{ uniformize} X \\ p\downarrow & \Delta \leq \mathrm{Aut}(\mathcal{H}),\ \ \Delta/\Gamma \cong G \\ X\ \ G & \mathrm{Aut}(\mathcal{H}) = \mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}^+(\mathcal{H}) \end{array}$$

Γ and Δ are Fuchsian groups - discrete and cocompact subgroups of the group of orientation preserving isometries of the hyperbolic plane $\mathcal{H}$.

Summing up

Given a compact Riemann surface X there exists a fuchsian surface group Γ so that $X \cong \mathcal{H}/\Gamma$

Having a Riemann surface represented as above a finite group G is a group of its conformal automorphisms if and only if $G \cong \Delta/\Gamma$ for some other Fuchsian group Δ.

$$|\Gamma| \cong \pi_1(X) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \sqcap [\alpha_i, \beta_i] \rangle$$
$$|\Delta| \cong \langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \delta_1, \ldots, \delta_r \mid \delta_i^{m_i}, \sqcap \delta_i \sqcap [\alpha_i, \beta_i] \rangle$$

There are known necessary and sufficient conditions for abstract groups with the above presentations to be realized as Fuchsian groups (recall: discrete and cocompact groups of orientation preserving isometries of hyperbolic plane $\mathcal{H}$).

Bordered and non-orienatble surfaces

A similar approach for bordered and non-orienatble surfaces is also possible. This time however

• the analytic structure is replaced by a dianalytic which roughly speaking differ from it by the fact that complex conjugation $z \mapsto \bar{z}$ is involved for transition maps between charts and

• the folding map $a + bi \mapsto a + |b|i$ is allowed for local form of maps between (borderd) surfaces

• the role of Fuchsian groups play more general non-euclidean crystallographic groups which are discrete and cocompact groups of isometries hyperbolic plane $\mathcal{H}$ including orientation reversing ones.

On the orders of finite groups of self homeomorphisms

[Schwartz] A group of self homeomorphisms of a closed orientable or not topological surface of topological genus $g \geq 2$ has at most $84(g-1)$ elements.

[C.L.May] A group of self homeomorphisms of a bordered topological surface of algebraic genus $p \geq 2$ has at most $12(p-1)$ elements, where $p = \varepsilon g + k - 1$.

Fixed points on closed surfaces

[Wimann-Harvey] The order of a single self homeomorphism of a compact orientable surface of genus $g \geq 2$ is not bigger than $4g + 2$, this bound is attained for arbitrary g ans if this is so, then it has one fixed point.

[Szemberg 1991] If a single self homeomorphism has at least 2 fixed points then the Wimann bound can be strengthened to $4g$ and again this bound is attained for all $g \geq 2$.

Fixed points on closed surfaces

[Farkas, Kra] A single self homeomorphism of a compact closed orientable topological surface of genus $g \geq 2$, having $q \geq 3$ fixed points has order not exceeding

$$M = M(g, q) = 2g/(q-2) + 1$$

and this bound is attained for arbitrary integers $g \geq 2$ and $q \geq 3$ for which $M = M(g, q)$ is an integer i.e, there exists a closed orientable topological surface of genus g and its self homeomorphism having q fixed points and order M.

Fixed points on bordered surfaces

[Corrales, Gamboa, Gromadzki 1999] A periodic self-homeomorphism of a compact bordered topological surface of algebraic genus $p \geq 2$, having $q \geq 2$ fixed points has order not greater than $N = p/(q-1)+1$.

[Corrales, Gamboa, Gromadzki 1999] If the above bound N is attained, then the corresponding surface X is orientable, the self homeomorphism preserves the orientation of X, the number k of connected components of ∂X is a divisor of N, and $q \equiv k \pmod 2$ if N is even.

Oikawa and Arakawa results

A G-invariant subset A of a Riemann compact topological surface of cardinality $< |G|$ is said to be essential and if A has not essential invariant subsets then it is said to be irreducible.

[K. Oikawa 1956] A finite group of self-homeomorphisms of a closed orientable surface of genus $g \geq 2$ and with an essential irreducible invariant subset of order k has at most $12(g-1)+k$ elements.

[T. Arakawa 2000] A finite group of self-homeomorphisms of a closed orientable surface of genus $g \geq 2$ and with two essential irreducible invariant subsets of cardinalities k, l has at most $8(g-1)+k+4l$ elements.

[T. Arakawa 2000] A finite group of self-homeomorphisms of a closed orientable surface of genus $g \geq 2$ and with three essential irreducible invariant subsets of orders k, l, m has at most $2(g-1)+k+l+m$ elements.

On Oikawa and Arakawa results

[E. Bujalance, G. Gromadzki 2011] A finite group of self-homeomorphisms of a closed orientable surface of genus $g \geq 2$ and with two essential irreducible invariant subsets of cardinalities k, l has either less than $2(g-1)+k+l$ elements or precisely

$$\frac{M}{M-1}\left(2(g-1)+k+l\right)$$

elements for some $M \geq 2$. Furthermore given $M \geq 2$ there are infinitely many values of g for which this bound is attained

[Corollary] A finite group of self-homeomorphisms of a closed orientable surface of genus $g \geq 2$ and with two essential irreducible invariant subsets of cardinalities k, l has at most $4(g-1)+2k+2l$ and this bound is attained for infinitely many values of g.

[Corollary] The bound of Arakawa $8(g-1)+k+4l$ is never attained.

More than 3 invariant subsets

[E. Bujalance, G. Gromadzki 2011] Let G be a finite group of self-homeomorphisms of a closed orientable surface of genus $g \geq 2$ having n invariant irreducible subsets of cardinalities $q_1 \leq \ldots \leq q_n$, $s \geq 4$ of them being essential. Then each q_i divides $|G|$ and

$$|G| \leq \frac{2}{s-2}(g-1) + \frac{q_1 + \ldots + q_s}{s-2}.$$

Conversely, for each s these bounds are attained for infinitely many values of g.

Nonorientable closed surfaces

[E. Bujalance, G. Gromadzki 2011] A finite group of self-homeomorphisms of a non-orientable closed surface of genus $g \geq 3$ with n invariant irreducible subsets of cardinalities $q_1 \leq \ldots \leq q_n$ first $s \geq 3$ of which are the essential has at most

$$\frac{2}{s-2}(g - 2 + q_1 + \ldots + q_s)$$

elements and this bound is attained for infinitely many g.

Precise bounds for $s = 1, 2$ are also known.

Bordered orientable compact surfaces

[E. Bujalance, G. Gromadzki 2011] A finite group of self-homeomorphisms (including orientation reversing ones) of a bordered orientable surface of algebraic genus $p \geq 2$ with s invariant subsets of interior points of cardinalities $q_1, \ldots, q_s$ has at most

$$\frac{1}{s-1}(p-1) + \frac{1}{s-1}(q_1 + \ldots + q_s)$$

elements.

Bordered nonorientable compact surfaces

[E. Bujalance, G. Gromadzki 2011] A finite group of self-homeomorphisms of a bordered non-orientable surface of algebraic genus genus $p \geq 2$ with $s \geq 2$ invariant subsets of interior points of cardinalities $q_1, \ldots, q_s$ has at most

$$\frac{1}{s-1}(p-1) + \frac{1}{s-1}(q_1 + \ldots + q_s)$$

elements while if it has also having t invariant subsets of the set of boundary components of cardinalities $p_1, \ldots, p_t$, where now $t, s \geq 1$ ar so that $t + s \geq 3$ has at most

$$\frac{2}{s+t-2}(g-2) + \frac{2}{s+t-2}(p_1 + \ldots + p_s + q_1 + \ldots q_t)$$

elements

The bibliography

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Announcement

Workshop on low dimensional conformal structures and their groups (Gdańsk, 27-29 June 2012)

http://mat.ug.edu.pl/conformal/