On imprimitive rank 3 permutation groups

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Rank of a permutation group

G transitive permutation group on Ω

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The rank of G is the number of orbits of G on $\Omega \times \Omega$.

- $\{(\alpha, \alpha) \mid \alpha \in \Omega\}$ is one orbit.
- G has rank 2 if and only if $\{(\alpha, \beta) \mid \alpha \neq \beta\}$ is an orbit, that is, if and only if G is 2-transitive.

Suborbits

There is a one-to-one correspondence between orbits of G_α on Ω and orbits of G on $\Omega \times \Omega$ given by

$$\beta^{G_\alpha} \longleftrightarrow (\alpha, \beta)^G$$

Hence the rank of G is also the number of orbits of G_α on Ω.

Wreath Products

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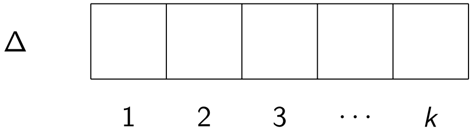
The group $H \,\mathrm{wr}\, K$ has two natural actions:

- imprimitive action: on $\Delta \times \{1, \ldots, k\}$ where

$$(\delta, i)^{(h_1, \ldots, h_k)\sigma} = (\delta^{h_i}, i^{\sigma})$$



Note that G permutes the sets $\Delta \times \{i\}$.

Wreath Products II

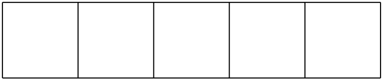
- product action: on Δ^k, where

$$(\delta_1, \ldots, \delta_k)^{(h_1, \ldots, h_k)} = (\delta_1^{h_1}, \ldots, \delta_k^{h_k})$$

$$(\delta_1, \ldots, \delta_k)^{\sigma} = (\delta_{1^{\sigma^{-1}}}, \ldots, \delta_{k^{\sigma^{-1}}})$$

Systems of imprimitivity

A transitive group is called imprimitive if it preserves some nontrivial partition of Ω.



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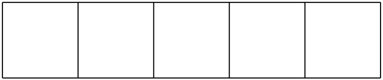

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The stabiliser of a partition into b parts of size a in S_{ab} is

$$S_a \operatorname{wr} S_b$$

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- $G^{\mathcal{B}}$, the subgroup of $\mathrm{Sym}(\mathcal{B})$ induced by G, is transitive.
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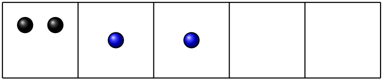
- G_B^B, the subgroup of $\mathrm{Sym}(B)$ induced by the setwise stabiliser G_B, is transitive;
- action of G is isomorphic to a subgroup of $G_B^B \mathrm{wr}\, G^{\mathcal{B}}$

2-transitive groups

All 2-transitive groups are primitive.



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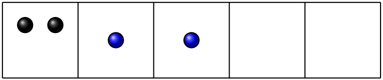

Burnside's Theorem: A 2-transitive group is either:

- almost simple, that is $T \leqslant G \leqslant \mathrm{Aut}(T)$ with T a nonabelian simple group, or
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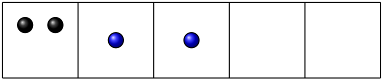

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All finite 2-transitive groups were classified as a consequence of the Classification of Finite Simple Groups.

Rank 3 groups

Study goes back to Donald G. Higman in 1964.

Some examples:

- S_n acting on 2-subsets.
- $\mathrm{PGL}_n(q)$ acting on 2-subspaces
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- $S_n \,\mathrm{wr}\, S_2$ acting in product action on Δ^2, where $|\Delta| = n$,
- $S_a \,\mathrm{wr}\, S_b$ acting imprimitively on ab points.

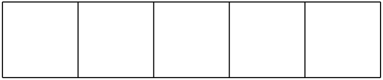
Primitive rank 3 groups

All primitive rank 3 groups have been classified. They are either

- almost simple (Bannai, Kantor-Liebler, Liebeck-Saxl)
- a subgroup of $\mathrm{AGL}(d, p)$ (Liebeck)
- a subgroup of $H \operatorname{wr} S_2$ acting on Δ^2, where H is an almost simple 2-transitive group on Δ

Imprimitive rank 3 groups

Recall that an imprimitive group G is a subgroup of $G_B^B \operatorname{wr} G^{\mathcal{B}}$.



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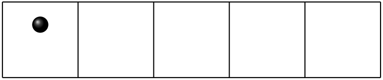

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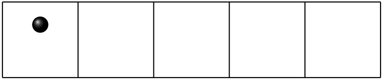

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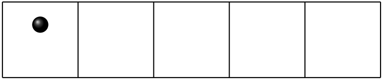

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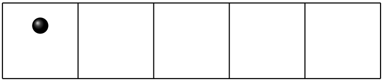

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Which subgroups of $H \,\mathrm{wr}\, K$ have rank 3?

A characterisation

Theorem

Suppose that G is an imprimitive group with

- G_B^B *a 2-transitive almost simple group with socle T*
- $G^{\mathcal{B}} \leqslant S_k$ *is 2-transitive.*

Then G has rank 3 if and only if one of the following holds:

1. $T^k \leqslant G$
2. *G is quasiprimitive and rank 3*
3. *$k = 2$ and $G = M_{10}$, $\mathrm{PGL}(2, 9)$ or $\mathrm{Aut}(A_6)$ acting on 12 points;*
4. *$k = 2$ and $G = \mathrm{Aut}(M_{12})$ acting on 24 points.*

Quasiprimitive groups

A permutation group is called quasiprimitive if every nontrivial normal subgroup is transitive.

Every primitive group is quasiprimitive.

If G is quasiprimitive and imprimitive then it acts faithfully on any system of imprimitivity $\mathcal{B}$.

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Lemma

An imprimitive, block faithful rank 3 group is almost simple.

Theorem

All imprimitive, block faithful almost simple groups such that $G^{\mathcal{B}}$ and G_B^B are 2-transitive have been classified.

Theorem

A quasiprimitive rank 3 group is either primitive or listed in the table.

G	k	m	G_B^B	extra conditions
M_{11}	11	2	C_2	
$G \geqslant \mathrm{PSL}(2,q)$	$q+1$	2	C_2	$q \equiv 1 \pmod 4$ plus other conditions on G
$G \geqslant \mathrm{PSL}(a,q)$	$\frac{q^a-1}{q-1}$	m	$\mathrm{AGL}(1,m)$	$a \geq 3$, m prime plus other conditions
$\mathrm{PGL}(3,4)$	21	6	$\mathrm{PSL}(2,5)$	
$\mathrm{P\Gamma L}(3,4)$	21	6	$\mathrm{PGL}(2,5)$	
$\mathrm{PSL}(3,5)$	31	5	S_5	
$\mathrm{PSL}(5,2)$	31	8	A_8	
$\mathrm{P\Gamma L}(3,8)$	73	28	$\mathrm{Ree}(3)$	
$\mathrm{PSL}(3,2)$	7	2	C_2	
$\mathrm{PSL}(3,3)$	13	3	S_3	