# One-regular graphs

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Symmetries of Discrete Objects
Queenstown, New Zealand, 13-17 February 2012

## Outline

## Aim of this report

In this report I first give a brief survey of one-regular graphs. Then I will talk about a conjecture, that is the existence of one-regular 3-valent graphs of order $4m$ for an odd integer $m$, which was answered by Conder and the author.

## Regular permutation groups

- Let $G$ be a permutation group on $\Omega$, that is, $G \leq S_{\Omega}$.
- $G$ is transitive on $\Omega$: for any two points in $\Omega$ there is a permutation in $G$ mapping one to the other.
- $G$ is regular on $\Omega$: for any two points in $\Omega$ there is one and only one permutation in $G$ mapping one to the other, that is, only the identity element in the transitive subgroup fixes a point.
- A regular permutation group is 'the smallest possible transitive group'.

## Notation for graphs

- $X$: a simple graph (no loops or multiple edges).
- $V(X)$, $E(X)$: the vertex set and the edge set.
- The automorphism group $\mathrm{Aut}(X)$ of a graph $X$: the group of all permutations on $V(X)$ preserving the adjacency of $X$, that is, mapping an edge to an edge.
- $X$ is vertex-transitive or edge-transitive: $\mathrm{Aut}(X)$ is transitive on $V(X)$ or $E(X)$, respectively.

## Notation for graphs

- $s$-arc: an $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices s.t. $\{v_i, v_{i+1}\} \in E(X)$, $v_{i-1} \neq v_{i+1}$.
- $s$-arc-transitive: $\mathrm{Aut}(X)$ acts transitively on the set of $s$-arcs in $X$.
- 0-arc-transitive: vertex-transitive.
- 1-arc-transitive: arc-transitive or symmetric
- $s$-arc-regular graph: $\mathrm{Aut}(X)$ acts regularly on the set of $s$-arc of $X$.
- one-regular graph: 1-arc-regular graph.

## Cayley graph

$G$: a finite group, $S \subset G$, $1 \notin S$, $S = S^{-1} = \{s^{-1} \mid \in S\}$.

- Cayley graph $\mathrm{Cay}(G, S)$: vertex set $V = G$, edge set $E = \{(g, sg) \mid g \in G, s \in S\}$
- $\mathrm{Cay}(G, S)$ is connected $\Leftrightarrow G = \langle S \rangle$.
- Right regular representation $R(G)$ of $G$: the permutation group $\{R(g) \mid g \in G\}$ on $G$, where $R(g) : x \mapsto xg$, $\forall x \in G$ is a permutation on $G$. Clearly, $R(G) \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$, acting regularly on $V(X)$.
- Characterization: A graph $X$ is a Cayley graph on $G \Leftrightarrow \mathrm{Aut}(X)$ has a regualr subgroup isomorphic to $G$.

## Basic facts about one-regular graphs

- If an $s$-arc-regular graph is not connected, it must be a union of one vertex and a connected $s$-arc-regular graph.
- A 2-valent (regular) graph is one-regular if and only if it is a cycle $C_n$ for some positive integer $n \geq 3$. On the other hand, $C_n$ is $s$-regular for any $s \geq 2$.
- When one consider one-regular graph, it is supposed that the graph is connected and has valency greater than 2.
- Some examples of cubic $s$-regular graphs: the 2-regular complete graph $K_4$, the 2-regular three dimensional hypercube $Q_3$, the 3-regular Petersen graph $O_3$.

## One-regular graphs with valency greater than 4

- One may easily obtain a classification of one-regular graphs of prime order by Burnside Theorem. (also see [1, 2]).
- Cheng and Oxley in [3] give a classification of one-regular graphs of order twice a prime.
- Kwak et al [21] Constructed infinitely many one-regular graphs of valency $4k$.
- Kwak et al [20] constructed an infinite family of one-regular Cayley graphs on dihedral groups of any even valency.

# One-regular graphs with valency greater than 4

- Kwak et al 2008 [18] constructed an infinite family of one-regular Cayley graphs on dihedral groups of any prescribed valency. In particular, a classification of one-regular Cayley graphs on a dihedral group of valency 5 can be reduced.
- Feng and Li [10] classified one-regular Cayley graphs of prime valency on dihedral groups, and as a result, one-regular graphs of square free order of prime valency were classified.
- Infinitely many one-regular Cayley graphs of valency 6 on dihedral groups were constructed by Hwang, Kwak and Oh [19, 27].

## One-regular graphs with valency 4

- Hwang, Kwak and Oh [19, 27] constructed infinitely many tetravalent one-regular Cayley graphs on dihedral groups.
- Wang, Xu and Zhou [29, 30] classified one-regular Cayley graphs of valency 4 on dihedral groups.
- Note that Du, Malnič and Marušič [8] classified 2-arc-transitive Cayley graphs on dihedral groups.
- Xu [33] give a classification of tetravalent one-regular circulant graphs.
- Xu and Xu [31] give a classification of tetravalent one-regular Cayley graphs on abelian groups.

# One-regular graphs with valency 4

- All tetravalent one-regular graphs of order p or pq are circulant, and a classification of such graphs can be easily deduced from [32].
- Zhou and Feng [35, 37] classified tetravalent one-regular graphs of order $2pq$, where $p$ and $q$ are primes.
- An infinite family of tetravalent one-regular Cayley graphs on alternating groups was constructed by Marušič in [22].
- An infinite family of infinite one-regular graphs of valency 4 was constructed by Malnič et al [23].

## One-regular graphs with valency 3

- The first one-regular cubic graphs was constructed by Frucht in [28].
- Conder and Dobcsányi [5] classified one-regular ($s$-regular) cubic graphs of order up to 768.
- Marušič and Pisanski [24] classified one-regular ($s$-regular) Cayley graphs of valency 3 on a dihedral group.
- Zhou and Feng [36] classified cubic one-regular graphs of square-free order.
- Kutnar and Marušič [17] classified one-regular ($s$-regular) Cayley graphs of valency 3 on a generalized dihedral group.

# One-regular graphs with valency 3

- Feng and Kwak [14] constructed an infinite family of cubic one-regular Cayley graphs on alternating groups.
- Du and Wang [9] proved that there is no cubic one-regular Cayley graphs on $\mathrm{PSL}(2, p)$, where $p \geq 5$ is a prime.
- Feng, Kwak, et al [11, 16, 12, 15, 13] classified cubic one-regular ($s$-regular) graphs of order $2p^2$, $2p^3$, $mp$ and $mp^2$ for $m = 4, 6, 8, 10$, where $p$ is a prime.
- Oh [25, 26] classified cubic one-regular ($s$-regular) graphs of order $14p$ and $16p$.

## A conjecture on one-regular cubic graphs

- By checking all cubic one-regular graphs discovered before, there is no cubic one-regular graphs of order 4 times an odd integer. Then a natural conjecture follows:
- Conjecture [36]: There is no cubic one-regular graphs of order $4m$ for any odd integer $m$.
- However, the conjecture is not true. Recently, Conder and Feng [4] answered the above conjecture negatively by proving the following results.

- Theorem 1: Let $X$ be a one-regular cubic graph of order $4m$ where $m$ is odd. Then $X$ is a normal cover of a base graph $Y$, where $Y$ has an arc-regular group of automorphisms that is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{PSL}(2,q))$ containing $\mathrm{PSL}(2,q)$ for some odd prime-power $q$.

- To state the second result, we need some notation. Let $p$ be an odd prime and let $K = \mathrm{GF}(p^3)$ be the field of order $p^3$. Denote by $\alpha$ the Frobenius automorphism of $K$: $\alpha : x \mapsto x^p$. For any matrix $M \in \mathrm{SL}(2,K)$, denote by $\overline{M}$ the image of $M$ under the natural homomorphism from $\mathrm{SL}(2,K)$ to $\mathrm{PSL}(2,K) = \mathrm{SL}(2,K)/Z(\mathrm{SL}(2,K))$.

- **Theorem 2**: For any element $t \in K$ such that $t^3$ lies outside the base field $F = \mathrm{GF}(p)$, let

$$U = \begin{pmatrix} 1 & -2t \\ t^{-1} & -1 \end{pmatrix}, \quad V = U^{\alpha} = \begin{pmatrix} 1 & -2t^{p} \\ t^{-p} & -1 \end{pmatrix},$$

$$W = V^{\alpha} = \begin{pmatrix} 1 & -2t^{p^2} \\ t^{-p^2} & -1 \end{pmatrix}.$$

Then

(1) the images $\overline{U}$, $\overline{V}$ and $\overline{W}$ generate $\mathrm{PSL}(2, K)$, and
(2) the Cayley graph $\mathrm{Cay}(\mathrm{PSL}(2, K), \{\overline{U}, \overline{V}, \overline{W}\})$ is a one-regular cubic graph.

## skeleton Proof of Theorem 1

- Let $A = \mathrm{Aut}(X)$, $P \in \mathrm{Syl}_2(A)$. Then $|A| = 3|V(X)| = 12m$ and $|P| = 4$, so $P \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- $P \cong \mathbb{Z}_4 \mapsto N_A(P)/C_A(P) \lesssim \mathrm{Aut}(P) \cong \mathbb{Z}_2$ (N/C theorem).
- $|N_A(P)/C_A(P)| = 2 \mapsto (P \leq C_A(P))$ $|N_A(P)|$ is divisible by $2 \times 4 = 8$, contradiction.
- $|N_A(P)/C_A(P)| = 1 \mapsto N_A(P) = C_A(P) \mapsto$ there is $T \trianglelefteq A$ such that $A = TP$ and $T \cap P = 1$ (Burnside), so $|T| = |A|/|P| = 12m/4 = 3m \mapsto (|V(X)| = 4m)$ $T$ has four orbits on $V(X) \mapsto T$ is semiregular on $V(X)$ (Lorimer), contradiction.

## skeleton Proof of Theorem 1

- Let $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $N$ be the largest normal subgroup of $A$ of odd order. Then $N$ has at least four orbits and $X$ is a normal cover of $X_N$ (Lorimer).
- Gorenstein-Walter theorem $\mapsto A/N \cong P$, or $A_7$, or a subgroup of $\mathrm{Aut}(\mathrm{PSL}(2, q))$ containing $\mathrm{PSL}(2, q)$ for some odd $q$
- $A/N \cong P \mapsto |N| = 3m$ and $N$ is semiregular on $V(X)$ (Lorimer), which is impossible.
- Clearly, $A/N \ncong A_7$ because $8 \nmid |A|$.
- $A/N \cong$ a subgroup of $\mathrm{Aut}(\mathrm{PSL}(2, q))$ containing $\mathrm{PSL}(2, q)$ for some odd $q$.

## skeleton Proof of Theorem 2

- Let $S = \{\overline{U}, \overline{V}, \overline{W}\}$, $G = \mathrm{PSL}(2, K)$, $X = \mathrm{Cay}(G, S)$, and $A = \mathrm{Aut}(X)$.
- The fact that $\langle \overline{U}, \overline{V}, \overline{W} \rangle = \mathrm{PSL}(2, K)$ is proved by considering maximal subgroups of $\mathrm{PSL}(2, q)$, which was first given by Dickson [6].
- To prove that $X$ is one-regular, it suffices to show that $A = R(G) \rtimes \mathrm{Aut}(G, S)$ and $\mathrm{Aut}(G, S) = \langle \overline{\alpha} \rangle$, where $\overline{\alpha}$ is the automorphism of $G$ induced by $\alpha$.
- Xu et al. [34] $\mapsto$ $A = R(G) \rtimes \mathrm{Aut}(G, S)$. Clearly, $|\mathrm{Aut}(G, S)| = 3$ or 6. The former implies $A = R(G) \rtimes \mathrm{Aut}(G, S)$. We only need to show that the latter cannot happen.

## skeleton Proof of Theorem 2

- Suppose $|\mathrm{Aut}(G,S)| = 6$. Let $B = \langle R(G), \overline{\alpha} \rangle$. Then $|A:B| = 2$ and $B \cong \mathrm{P\Sigma L}(2,K)$. Let $C = C_A(B)$, the centralizer of $B$ in $A$.
- $C \cap B = Z(B) = 1 \mapsto |C| = 1$ or 2. Note that $A_1 \cong S_3$.
- $|C| = 2 \mapsto A = B \times C \mapsto A_1 \cong A/R(G) \cong \mathbb{Z}_6$, contradiction.
- $|C| = 1 \mapsto A \lesssim \mathrm{Aut}(B)$ (N/C theorem) $\cong \mathrm{Aut}(\mathrm{P\Sigma L}(2,K)) \cong \mathrm{P\Gamma L}(2,K) \mapsto A \cong \mathrm{P\Gamma L}(2,K)$ (order).
- $A_1 \cong A/R(G) \cong \mathrm{P\Gamma L}(2,K)/\mathrm{PSL}(2,K) \cong \mathbb{Z}_6$, contradiction.

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Thanks!