A Classification of Orientable Regular Embeddings of Complete Multipartite Graphs

Shaofei Du

School of Mathematical Sciences
Capital Normal University
Beijing, 100048, China

.

.

Symmetries of Discrete Object
Conference and MAGAA Workshop
Queenstown, New Zealand
Feb 13-17, 2012 .

.

http://arxiv.org/abs/1202.1974 .
Joint Work with Mr Junyang Zhang

Content

$K_{m[n]}$: Complete multipartite graph,

m parts, each part contains n vertices,

any two vertices in different parts are adjacent

1. Definitions
2. Regular Embeddings of $K_{m[1]}$, $K_{2[n]}$, $K_{m[p]}$
3. Regular Embeddings of $K_{m[n]}$, $m \geq 3$, $n \geq 2$

Surfaces and Embeddings

Surface S: closed, connected 2-manifold;

Classification of Surfaces:
(i) Orientable Surfaces: S_g, $g = 0, 1, 2, \cdots$,
$v + f - e = 2 - 2g$

(ii) Nonorientable Surfaces: N_k, $k = 0, 1, 2, \cdots$,
$v + f - e = 2 - k$

Embeddings of a graph X in the surface is a continuous one-to-one function $i : X \to S$.

2-cell Embeddings: each region is homemorphic to an open disk.

Topological Map $\mathcal{M}$: a 2-cell embedding of a graph into a surface. The embedded graph X is called the *underlying graph* of the map.

Automorphism of a map $\mathcal{M}$: an automorphism of the underlying graph X which can be extended to self-homeomorphism of the surface.

Orientation-Preserving Automorphism of an orientable map $\mathcal{M}$: an automorphism of Preserving Orientation of the map

Automorphism group $\mathrm{Aut}(\mathcal{M})$: all the automorphisms of the map $\mathcal{M}$.

Orientation-preserving automorphisms group $\mathrm{Aut}^+\mathcal{M}$ of $\mathcal{M}$: all the oientation-preserving automorphism.

Flag: incident vertex-edge-face triple

Arc: incident vertex-edge pair

Remark: $\mathrm{Aut}(\mathcal{M})$ acts semi-regularly on the flags of X.

Remark: $\mathrm{Aut}^+(\mathcal{M})$ acts semi-regularly on the arcs of X.

Regularity of Maps

Regular Map: $\mathrm{Aut}(\mathcal{M})$ acts regularly on the flags.

Orientable Regular Map: $\mathrm{Aut}^+(\mathcal{M})$ acts regularly on the arcs.

Reflexible Map: Orientable Regular, admiting orientation-reversing automorphisms

Chiral Map: Orientable Regular, without any orientation-reversing automorphisms

Regular Map: 'Nonorientable Regular Map' + 'Reflexible Orientable Regular Map'

Orientable Regular Map: 'Reflexible Orientable Regular Map' + 'Chiral Orientable Regular Map'

Combinatorial Orientable Map:

graph $X = (V, D)$, with vertex set $V = V(X)$, dart (arc) set $D = D(X)$.

arc-reversing involution L: interchanging the two arcs underlying every given edge.

rotation R*:* cyclically permutes the arcs initiated at v for each vertex $v \in V(X)$.

Map $\mathcal{M}$ *with underlying graph* X*:*
the triple $\mathcal{M} = \mathcal{M}(X;\ R,\ L)$.

Remarks:

Monodromy group $\mathrm{Mon}(\mathcal{M}) := \langle R, L \rangle$ acts transitively on D.

Given two maps

$$\mathcal{M}_1 = \mathcal{M}(X_1;\ R_1,\ L_1),\ \mathcal{M}_2 = \mathcal{M}(X_2;\ R_2\, L_2),$$

Map isomorphism: bijection $\phi : D(X_1) \to D(X_2)$ such that

$$L_1\phi = \phi L_2, R_1\phi = \phi R_2$$

Automorphism ϕ of $\mathcal{M}$: if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$;

Automorphism group: $\mathrm{Aut}(\mathcal{M})$

Algebraic Orientable Maps:

Orientable Regular Map:

$G = \mathrm{Aut}(\mathcal{M}) = \langle r, l \rangle \cong \mathrm{Mon}(\mathcal{M}) = \langle R, L \rangle$

$\mathcal{M} = \mathcal{M}(G; r, l)$

$G_v = \langle r \rangle$.

1. Complete graphs $K_{m[1]}$:

Orientrable:

N.L. Biggs, Classification of complete maps on orientable surfaces, *Rend. Mat.* (6) **4** (1971), 132-138.
L.D. James and G.A. Jones, Regular orientable imbeddings of complete graphs, *J. Combin. Theory Ser. B* **39** (1985), 353–367.

Nonorientrable:

S. E. Wilson, Cantankerous maps and rotary embeddings of K_n, *J. Combin. Theory Ser. B* **47** (1989), 262–273.

m must be of order 3, 4 or 6.

2. Complete bipartite graphs $K_{2[n]}$:

Nonorientable Case:

J.H.Kwak and Y.S.Kwon, Classification of nonorientable regular embeddings complete bipartite graphs, *J. Combin. Theory, Ser. B* **101(2011) 191-205.**

Orientable Case:

Survey paper: G.A. Jones, Maps on surfaces and Galois groups, *Math. Slovaca* **47** (1997), 1-33.

General approach: R. Nedela, M. Škoviera and A. Zlatoš, Regular embeddings of complete bipartite graphs, *Discrete Math.* **258**(2002) 379-381.

Partial Results:

$n = pq$ J.H. Kwak and Y.S. Kwon
Regular orientable embeddings of complete bipartite graphs, *J. Graph Theory* **50**(2005), 105-122.

Reflexible maps:, J. H. Kwak and Y. S. Kwon
Classification of reflexible regular embeddings and self-Petrie dual regular embeddings of complete bipartite graphs, *Discrete Math.* **308**(2008) 2156-2166.

$(n, \phi(n)) = 1$*:* G.A. Jones, R. Nedela and M. Škoviera
G. A. Jones, R. Nedela and M. Škoviera, Complete bipartite graphs with a unique regular embedding, *J. Combin. Theory Ser. B* **98**(2008), 241-248.

Complete Classification:

$n = p^k$, *p is odd prime:* G.A. Jones, R. Nedela and M. Škoviera

Regular embeddings of $K_{n,n}$ where n is an odd prime power, *European J. Combin.* **28**(2007), 1863-1875.

$n = 2^k$, S.F. Du, G.A.Jones, J.H. Kwak, R. Nedela and M. Škoviera,

Regular embeddings of $K_{n,n}$ where n is a power of 2. I: Metacyclic case, *European J. Combin.* **28** (2007), 1595-1608.

Regular embeddings of Kn,n where n is a power of 2. II: Nonmetacyclic case, European J. Combin. 31(7), 1946-1956. 2010.

Any n: G.A. Jones,

Regular embeddings of complete bipartite graphs: classification and enumeration,*Proc. London Math. Soc.* **101**(2010), 427-453.

3. Complete multipartite graphs $K_{m[p]}$:

S. F. Du, J. H. Kwak, R. Nedela,
Regular embeddings of complete multipartite graphs, *European J. Combin.* **26**(2005), 505-519.

Theorem

Classification Theorem: For $m \geq 3$ and $n \geq 2$, let $K_{m[n]}$ be the complete multipartite graph with m parts, while each part contains n vertices. Let $\mathcal{M}$ be an orientable regular embedding of $K_{m[n]}$. Let $G = \mathrm{Aut}(\mathcal{M})$, H the subgroup of G fixing each part setwise and $P \in Syl_p(G)$. Then $\mathcal{M}$ is isomorphic to one of the following five families of maps $\mathcal{M}_i$, where $1 \leq i \leq 5$:

[1] $m = p \geq 3$, $n = p^e$: p a prime, $e \geq 1$,

$H \cong \mathbb{Z}_p^2$ for $e = 1$, H is nonabelian for $e \geq 2$, $\mathrm{Exp}\,(P) = p^{e+1}$:

$G_1(p,e) = \langle a, c | a^{p^e(p-1)} = c^{p^{e+1}} = 1, c^a = c^r \rangle \cong Z_{p^{e+1}} : Z_{\phi(p^e)}$,
where $\mathbb{Z}^*_{p^{e+1}} = \langle r \rangle$;

$\mathcal{M}_1(p,e,j) = \mathcal{M}(G_1; a^j, a^{\frac{p^e(p-1)}{2}} c)$, where $j \in \mathbb{Z}^*_{p^e(p-1)}$.

[2] $m = n = p \geq 3$, p a prime, $H \cong \mathbb{Z}_p^2$ and $\mathrm{Exp}\ (P) = p$:

$G_2(p) = \langle w, z, c, g \mid \langle w, z\rangle \cong \mathbb{Z}_p^2, c^p = g^{p-1} = 1, c^g = c^t, w^c = wz, z^c = z, w^g = w, z^g = z^t\rangle \cong Z_p^2 : (Z_p : Z_{p-1})$,
where $\mathbb{Z}_p^* = \langle t\rangle$;

$\mathcal{M}_2(p, j) = \mathcal{M}(G_2; (wg)^j, (wg)^{\frac{p-1}{2}}c)$, where $j \in \mathbb{Z}_{p-1}^*$.

[3] $m = p = 3$, $n = k3^e$ for $3 \nmid k$, either $e = 0, 1$ and $k \geq 2$ or $e \geq 2$, H is abelian:

$G_3(k, e, l) = \langle a, b \mid a^{2 \cdot 3^e k} = b^2 = 1, a^2 = x, x^b = y, [x, y] = 1, y^a = x^{-1}y^{-1}, (ab)^3 = x^{3^{e-1}kl}y^{-3^{e-1}kl} \rangle$,
where $l = 0$ for $e = 0$; and $l = 0, 1$ for $e \geq 1$;

$\mathcal{M}_3(k, e, l, j) = \mathcal{M}(G_3; a^j, b)$, where $(l, j) = (0, 1), (1, \pm 1)$.

[4] $m = p = 3$, $n = k3^e$ for $3 \nmid k$, $k \geq 2$, $e \geq 2$,
H is nonabelian, $\mathrm{Exp}\,(P) = 3^{e+1}$:

$G_4(k,e) = \langle a, b \mid a^{2\cdot 3^e k} = b^2 = 1, c = a^{3^e} b, a^{2\cdot 3^e} = x_1, x_1^b = y_1, [x_1, y_1] = 1, y_1^a = x_1^{-1} y_1^{-1}, c^{3^{e+1}} = 1, c^a = c^2 x_1^u y_1^{\frac{u-1}{2}} \rangle$,
where $u3^e \equiv 1 (\mathrm{mod}\ k)$;

$\mathcal{M}_4(k,e,j) = \mathcal{M}(G_4; a^j, b)$, where $j \in \mathbb{Z}^*_{2k\cdot 3^e}$

$\mathcal{M}_4(k,e,j_1) \cong \mathcal{M}_4(k,e,j_2)$ if and only if $j_1 \equiv j_2 (\mathrm{mod}\ 2\cdot 3^e)$.

[5)] $m = p = 3$, $n = 9k$ for $3 \nmid k$, H is nonabelian, $\mathrm{Exp}\ (P) = 9$:

$G_5(k, l) = \langle a, b \mid a^{18k} = b^2 = 1, a^2 = x, x^b = y, [x, y] = x^{3k}y^{-3k}, y^a = x^{-1}y^{-1}, (ab)^3 = x^{3lk}y^{-3lk}\rangle$,
where $l = 0$ or ± 1;

$\mathcal{M}_5(k, l, j) = \mathcal{M}(G_5; a^j, b)$, where $j = \pm 1$.

Table 1:

Enumerations of the resulting maps

Maps	Number	Reflexible or Chiral	Type $\{s, t\}$ s-gon, valency t
$\mathcal{M}_1(p, e, j)$	$\phi(p^e)\phi(p-1)$	C	$\{p^e(p-1), p^e(p-1)\}$ if $p \equiv 1 \pmod 4$, $\{\frac{p^e(p-1)}{2}, p^e(p-1)\}$ if $p \equiv 3 \pmod 4$
$\mathcal{M}_2(3, 1)$	1	R	$\{3, 6\}$
$\mathcal{M}_2(p, j), p \geq 5$	$\phi(p-1)$	C	$\{p(p-1), p(p-1)\}$ if $p \equiv 1 \pmod 4$, $\{\frac{p(p-1)}{2}, p(p-1)\}$ if $p \equiv 3 \pmod 4$
$\mathcal{M}_3(k, e, 0, 1)$	1	R	$\{3, 2 \cdot 3^e k\}$
$\mathcal{M}_3(k, e, 1, \pm 1)$	2	C	$\{9, 2 \cdot 3^e k\}$
$\mathcal{M}_4(k, e, j)$	$2 \cdot 3^{e-1}$	C	$\{3^{e+1}, 2 \cdot 3^e k\}$
$\mathcal{M}_5(k, l, j)$	6	C	$\{3, 18k\}$ if $l = 0$, $\{9, 18k\}$ if $l = \pm 1$

Table 2:

Total numbers of regular embeddings of $K_{m[n]}$, $n = 3^e k, 3 \nmid k$

m	n	Reflexible	Chiral	Total
3	$k \geq 2$	1	0	1
	$3k, k \geq 1$	1	2	3
	$9k, k \geq 1$	1	14	15
	$3^e k, k \geq 1,$ $e \geq 3$	1	$2 \cdot 3^{e-1}+2$	$2 \cdot 3^{e-1} + 3$
$p \geq 5$	p	0	$p\phi(p-1)$	$p\phi(p-1)$
	$p^e (e \geq 2)$	0	$\phi(p^e)\phi(p-1)$	$\phi(p^e)\phi(p-1)$

Outline of Proof

1. Isobicyclic group H:

$H = \langle x\rangle\langle y\rangle$, where $|x| = |y| = n$, $\langle x\rangle \cap \langle y\rangle = 1$ and $x^\alpha = y$ for an involution $\alpha \in \mathrm{Aut}(H)$.

$n = p^k$ and p is odd prime: H is metacyclic, Huppert

$n = 2^k$: metacyclic $+$ 3 nonmetacyclic groups

Du, Jones, Kwak, Nedela and Škoviera

Also, Janko, *Israel J. Math.* **166** (2008), 313–347.

Lemma

Let (H, x, y) be a n-isobicyclic triple. Then H has a characteristic series

$$1 = H_0 < H_1 < \cdots < H_l = H$$

of subgroups $H_i = H^{s_i} = \langle x^{s_i} \rangle \langle y^{s_i} \rangle$ with $H_i/H_{i-1} \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$ for all $i \in [l]$, where $p_1 \geq \cdots \geq p_l$ are the prime divisors of n and $s_i = n/(p_1 \cdots p_i)$.

Lemma

Suppose that (H, x, y) is a n-isobicyclic triple and p is the maximal prime divisor of n. Let $L = H^{n/p}$. Then $H/C_H(L)$ is an isobicyclic group.

2. Key Theorems:

Theorem

Let $\mathcal{M}$ be a regular embedding of $K_{m[n]}$, where $m \geq 4$ and $n \geq 2$, and let H be the kernel of $G = \mathrm{Aut}(\mathcal{M})$ on the set of m parts. Then

1. $m = p$ and $n = p^e$ for some prime $p \geq 5$;

2. $Z(G) = 1$ and H is a n-isobicyclic group.

Proof :

Step 1: m is a prime power, $\overline{G} \cong \mathrm{AGL}(1, m)$ and H is a n-isobicyclic group.

Step 2: $C_G(H_i) = C_H(H_i)$ and $C_{G/H_i}(H/H_i) = Z(H/H_i)$, where $H_i = H^{s_i}$ for $s_i = n/(p_1 \cdots p_i)$ and $n = p_1 \cdots p_l$ where $p_1 \geq \cdots \geq p_l$ are primes.

Step 3: m is a prime, $\overline{G} \cong \mathrm{AGL}(1, p)$

Step 4: Show that $n = p^e$ for some e. $\square$

Theorem

Suppose $m = 3$ *and* $n = 3^e k$ *with* $e \geq 0$ *and* $3 \nmid k$*. Then* $H = Q \times K$*, where* Q *is a* 3^e*-isobicyclic group and* K *is an abelian* k*-isobicyclic group.*

Proof :

Step 1: Show that H is nilpotent.
Step 2: Each Hall $3'$-subgroup of H is abelian.

Subclass 1: $m = p \geq 3$ and $n = p^e$:

$G/H \cong Z_p : Z_{p-1}$, $H = Z_{p^e} Z_{p^e}$, isobicyclic, metacyclic

1.1 $\mathrm{Exp}\ (P) = p^{e+1}$:
$G_1(p,e) = \langle a, c | a^{p^e(p-1)} = c^{p^{e+1}} = 1, c^a = c^r \rangle$

1.2 $\mathrm{Exp}\ (P) = p^e$, H is nonabelian:

$G_5(1,l) = \langle a, b \mid a^{18} = b^2 = 1, a^2 = x, x^b = y, [x,y] = x^3 y^{-3}, y^a = x^{-1} y^{-1}, (ab)^3 = x^{3l} y^{-3l} \rangle$, where $l = 0$ or ± 1;

1.3 $\mathrm{Exp}\ (P) = p^e$, H is abelian:

$G_2(p) = \langle w, z, c, g \mid \langle w, z \rangle \cong \mathbb{Z}_p^2, c^p = g^{p-1} = 1, c^g = c^t, w^c = wz, z^c = z, w^g = w, z^g = z^t \rangle \cong Z_p^2 : (Z_p : Z_{p-1})$

$G_3(1,e,l) = \langle a, b \mid a^{2 \cdot 3^e} = b^2 = 1, a^2 = x, x^b = y, [x,y] = 1, y^a = x^{-1} y^{-1}, (ab)^3 = x^{3^{e-1} l} y^{-3^{e-1} l} \rangle$, $e \geq 2$, $l = 0, 1$

$G/H \cong S_3$, $H = Q \times K$, $Q = Z_{3^e} Z_{3^e}$, isobicyclic, metacyclic
$K = Z_k \times Z_k$

2.1 Q is abelian:

$G_3(k, e, l) = \langle a, b \mid a^{2 \cdot 3^e k} = b^2 = 1, a^2 = x, x^b = y, [x, y] = 1, y^a = x^{-1}y^{-1}, (ab)^3 = x^{3^{e-1}kl} y^{-3^{e-1}kl} \rangle$,
where $k \geq 2$, $l = 0$ for $e = 0$; and $l = 0, 1$ for $e \geq 1$.

2.2 Q is nonabelian, $\mathrm{Exp}\ (P) = 3^{e+1}$

$G_4(k, e) = \langle a, b \mid a^{2 \cdot 3^e k} = b^2 = 1, c = a^{3^e} b, a^{2 \cdot 3^e} = x_1, x_1^b = y_1, [x_1, y_1] = 1, y_1^a = x_1^{-1} y_1^{-1}, c^{3^{e+1}} = 1, c^a = c^2 x_1^u y_1^{\frac{u-1}{2}} \rangle$,
where $u3^e \equiv 1 (\mathrm{mod}\ k)$;

2.3 Q is nonabelian, $\mathrm{Exp}\ (P) = 3^e$

$G_5(k,l) = \langle a, b \mid a^{18k} = b^2 = 1, a^2 = x, x^b = y, [x,y] = x^{3k}y^{-3k}, y^a = x^{-1}y^{-1}, (ab)^3 = x^{3lk}y^{-3lk} \rangle$,
where $l = 0$ or ± 1.

Thank You Very Much !