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Finitely-presented groups in MAGMA

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Talk 1: **Finitely-presented groups**

This talk deals with ways of investigating groups defined by generators and relations, such as the triangle groups

$$\Delta(2,p,q) = \langle\, x, y, z \,|\, x^2 = y^p = z^q = xyz = 1 \,\rangle$$

associated with regular maps (of type $\{p,q\}$ in this case).

Given a group G with finite presentation $G = \langle\, X \,|\, R \,\rangle$, there are methods for

- finding the order of G (when this is finite)
- enumerating cosets of a finitely-generated subgroup of G
- obtaining a presentation for a finitely-generated subgroup
- finding all subgroups of up to a given index in G
- finding all quotients of G of up to a given order and all nilpotent quotients of G of up to a given class.

Summary of important functions for f.p. groups

- `ToddCoxeter(G,H)` ... gives a coset table for H in G
- `Order(G)` ... attempts to find the order of G
- `Rewrite(G,H)` ... finds a presentation for the subgroup H
- `LowIndexSubgroups(G,n)` ... finds subgroups of index $\leq n$
- `LowIndexNormalSubgroups(G,n)` ... finds normal subgroups of index $\leq n$
- `AbelianQuotient(G)` ... finds the abelianization G/G'
- `pQuotient(G,p,c)` ... finds p-quotients of G of class $\leq c$
- `NilpotentQuotient(G,c)` ... finds nilpotent quotients of G of class $\leq c$

Coset enumeration

Let $G = \langle X \mid R \rangle$, and let H be the subgroup generated by some finite set Y of words on the alphabet $X = \{x_1, \ldots, x_m\}$.

Methods exist for systematically enumerating the cosets Hg for $g \in G$. It is helpful to store these in a coset table, which shows the effect of multiplying each (numbered) coset Hg by a generator x_i or its inverse ${x_i}^{-1}$:

	x_1	x_2	$\ldots$	x_1^{-1}	x_2^{-1}	$\ldots$
1	2	3		4		
2				1		
3					1	
4	1					
$\vdots$						

Each relation from the defining presentation $\langle X \mid R \rangle$ for G forces pairs of cosets to be equal: $Hgr = Hg$ for all $g \in G$.

The same thing happens on application of each generator $y \in Y$ to the trivial coset H: $Hy = H$.

New cosets are defined (if needed), and all such coincidences are processed, until the coset table either 'closes' or has too many rows.

If the coset table closes with n cosets, then $|G : H| = n$. Moreover, the coset table gives us the natural permutation representation of G on the right coset space $(G : H)$.

If it does not close, then the index $|G : H|$ could be infinite, or just too large to be found (or it might even be small but the computation was not given enough resources).

Schreier coset graphs

Suppose the group G is generated by $X = \{x_1, x_2, \ldots, x_m\}$.

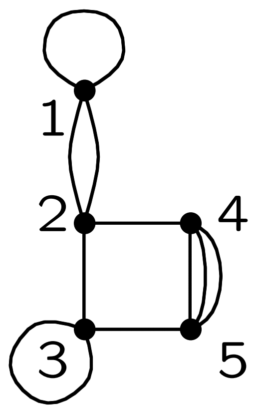
Given any transitive permutation representation of G on a set Ω of size n, we may form a graph with vertex-set Ω, and with edges of the form α — αx_i for $1 \leq i \leq m$.

Similarly, if H is a subgroup of index n in G, we may form a graph whose vertices are the right cosets of H and whose edges are of the form Hg — Hgx_i for $1 \leq i \leq m$.

These two graphs are the same when Ω is the right coset space $(G:H)$, and H is the stabilizer of a point of Ω. It is called the Schreier coset graph $\Sigma(G, X, H)$.

Schreier coset graphs (cont.)

The Schreier coset graph $\Sigma(G, X, H)$ gives a diagrammatic representation of the natural action of G on cosets of H, and hence is equivalent to the coset table, e.g. as follows:



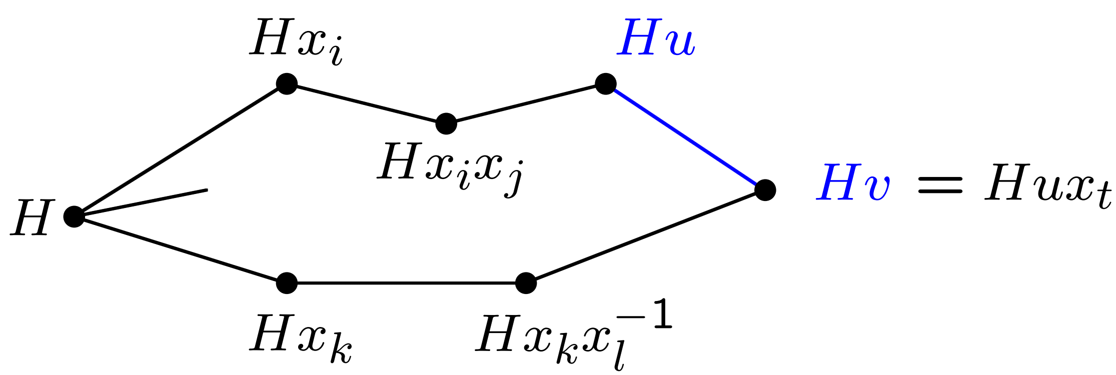
	x	y	x^{-1}	y^{-1}
1	2	1	2	1
2	1	3	1	4
3	3	5	3	2
4	5	2	5	5
5	4	4	4	3

when $x \mapsto (1,2)(4,5)$ and $y \mapsto (2,3,5,4)$

Some observations

1) A spanning tree for the coset graph Σ gives a Schreier transversal T for H in G

2) Edges of the coset graph not used in the spanning tree give a Schreier generating-set for H in G :



The edge Hu — Hv given by multiplication by x_i gives the Schreier generator $ux_iv^{-1} = ux_i(\overline{ux_i})^{-1}$ for H.

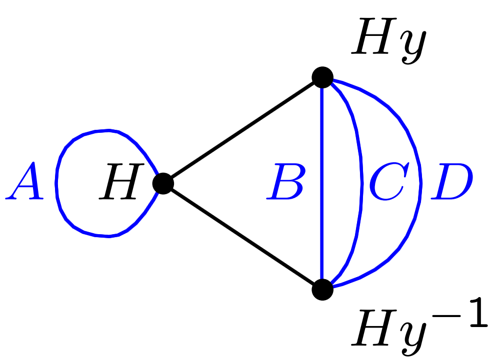
The Reidemeister-Schreier process

Given a finitely-presented group $G = \langle X \mid R \rangle$ and a subgroup H of finite index in G, Reidemeister-Schreier theory provides a method for obtaining a presentation for H (in terms of generators and relations):

1) Construct the coset graph — using the coset table
2) Take a spanning tree in the coset graph — this gives a Schreier transversal for H in G
3) Label the unused edges with Schreier generators
4) Apply each of the relators from R to each of the cosets in turn, to obtain the defining relations for H.

Example

Let $G = \langle x, y \,|\, x^2, y^3 \rangle$, and let H be the stabilizer of 1 in the permutation representation $x \mapsto (2,3)$, $y \mapsto (1,2,3)$:



Schreier generators

$A = x$

$B = y^3 \;(= 1)$

$C = yxy$

$D = y^{-1}xy^{-1}$

Relation $x^2 = 1$ gives new relations $A^2 = 1$ and $CD = 1$

Relation $y^3 = 1$ gives new relation $B = 1$

Thus H has presentation $\langle A, C \,|\, A^2 \rangle$ via $A = x$ and $C = yxy$.

Proving finitely-presented groups are infinite

There are several ways of proving a finitely-presented group $G = \langle X \mid R \rangle$ is infinite, with the help of MAGMA:

- Show the abelianisation G/G' is infinite
- Check to see if G has more generators than relations
- Find a subgroup of G with infinite abelianisation
- Construct an epimorphism onto a known infinite group

Note: the third of these depends on having a collection of subgroups to check — such as all subgroups of small index.

Low index subgroup methods

Again, let $G = \langle X \mid R \rangle$ be a finitely-presented group, and suppose we want to find all subgroups of small index in G.

Subgroups of index $\leq n$ can be found (up to conjugacy) by a systematic enumeration of coset tables with $\leq n$ rows.

The 'low index subgroups' algorithm starts with the identity subgroup and attempts to enumerate its right cosets. Then (or at any later stage) if more than n cosets are defined, all possible concidences between two cosets are considered.

This sets up a branching process for a backtrack search, which is guaranteed to complete (given sufficient time and memory), by Schreier's subgroup lemma!

Low index normal subgroups

Small homomorphic images of a finitely-presented group G can be found as the groups of permutations induced by G on cosets of subgroups of small index. This gives G/K where K is the core of H, but produces only images that have small degree faithful permutation representations.

Alternatively, the (standard) low index subgroups method can be adapted to produce only normal subgroups.

A new method was developed recently by Derek Holt and his student, which systematically enumerates the possibilities for the composition series of the factor group G/K, for any normal subgroup K of small index in G.

Some references

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