Tetravalent arc-transitive graphs of order $2pq$

Katja Berčič
with Mohsen Ghasemi, Primož Potočnik and Gabriel Verret

Institute Andrej Marušič
University of Primorska

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It the automorphism group of a graph acts transitively on the set of arcs, we say that the graph is *arc-transitive*.

We classify tetravalent arc-transitive graphs of order $2pq$ for distinct odd primes p and q.

Together with the previous work of Zhou and Feng who classified tetravalent arc-transitive graphs of order $4p$ and of order $2p^2$, this completes the classification of graphs of order $2pq$ for any two given primes p and q.

Definitions

1. *Whenever the automorphism group of a graph acts regularly on the set of arcs, we say that the graph is arc-regular.*
2. *A group is semisimple if it has no nontrivial abelian normal subgroups (equivalently, a trivial solvable radical).*
3. *We say that a graph Γ admits a group G if G is isomorphic to some subgroup of $\mathrm{Aut}(\Gamma)$.*

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The classification

Theorem

Let Γ be a tetravalent arc-transitive graph of order $2pq$ where p and q are distinct odd primes. Then one of the following holds:

(a) Γ *is an arc-regular graph: these have already been classified by Zhou and Feng;*

(b) Γ *is isomorphic to the lexicographic product $C_{pq}[2K_1]$ of the cycle C_{pq} and the edgeless graph on two vertices $2K_1$ (wreath graphs);*

(c) Γ *belongs to a (short) finite list of exceptions.*

Background

1. The classification of tetravalent arc-regular graphs of order $2pq$, where p and q are prime, by Zhou and Feng (2009).
2. The census of tetravalent 2-arc-transitive graphs of small order (Potočnik, 2009; up to 512 vertices; online up to 727 vertices).
3. The census of tetravalent arc-transitive graphs (Potočnik, Spiga, Verret, submitted; available online; up to 640 vertices).
4. It follows from the result of Praeger and Xu (1989) that if a tetravalent graph Γ admits an arc-transitive group G which contains a non-semiregular abelian normal subgroup N, then $\Gamma \cong C(2, r, s)$.

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In the proof, we use the abelian normal quotient method.

The normal quotient method: find a nontrivial intransitive normal subgroup and consider the quotient. Continue until you reach a quasiprimitive or a biquasiprimitive group (the base case). Reconstruct the graph.

The abelian normal quotient method: look for a nontrivial, abelian, minimal normal subgroup N and consider quotients until you reach a semisimple group. Note that if N is semiregular, the covers are normal.

The inductive process is easier when dealing with abelian normal subgroups; the base case is harder to solve in general. A strict order restriction makes the base for the abelian normal quotient method relatively simple.

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A sketch of the proof

- If the graph is arc-regular, refer to Zhou and Feng.
- If the automorphism group has an abelian, minimal normal subgroup N which is semiregular, consider the quotient graph (if it is not semiregular, the graph is isomorphic to $C_{pq}[2K_1]$).
  - If the quotient is arc-regular, then so is its normal cover.
  - If none of the above, reconstruct the graph from the base case or consider the next quotient.
- The base case: let p and q be distinct, odd primes and let Γ be a tetravalent graph of order $2pq$. If Γ admits an arc-transitive semisimple group G, then G has a unique minimal normal subgroup T, which is simple, and G embeds into $\mathrm{Aut}(T)$. Once we obtain a list of possible candidates for T, it can be show with a bit of help from Magma and the existing censuses that then Γ is isomorphic to one of the graphs in the table of exceptions.

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Main theorem: the exceptions

Γ	$\lvert V(\Gamma)\rvert$	$G_v^{\Gamma(v)}$	G	Description
$A_1[30,3]$	$2\cdot3\cdot5$	C_4	S_5	$L(\mathrm{cdc}(F_{10}))$
$A_2[30,1]$	$2\cdot3\cdot5$	C_2^2	S_5	$\mathrm{cdc}(L(F_{10}))$
$A_1[30,2]$	$2\cdot3\cdot5$	C_4	S_5	$D(F_{10})$
$A_1[30,5]$	$2\cdot3\cdot5$	C_2^2	S_5	
$A_1[42,3]$	$2\cdot3\cdot7$	C_4	$\mathrm{PSL}(2,7)$	$L(F_{28})$
		D_4	$\mathrm{PGL}(2,7)$	
$A_1[42,5]$	$2\cdot3\cdot7$	D_4	$\mathrm{PGL}(2,7)$	
$A_2[70,1]$	$2\cdot5\cdot7$	S_4	S_7	$\mathrm{cdc}(O(4))$
$A_2[110,1]$	$2\cdot5\cdot11$	A_4	$\mathrm{PGL}(2,11)$	$\mathcal{Y}(5,22;5,11)$
$A_2[182,2]$	$2\cdot7\cdot13$	A_4	$\mathrm{PGL}(2,13)$	
$A_2[506,1]$	$2\cdot11\cdot23$	A_4	$\mathrm{PSL}(2,23)$	
$A_2[506,2]$	$2\cdot11\cdot23$	A_4	$\mathrm{PSL}(2,23)$	
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T_{2162}	$2\cdot23\cdot47$	S_4	$\mathrm{PSL}(2,47)$	

If Γ admits an arc-transitive semisimple group, it belongs to the table above; if not, it is one of three other exceptions on 42, 182 or 506 vertices.

Thank you!