Computability and classification problems

Lecture 1:

1. Part 1: Computability and non-computability
2. Part 2: Computability, provability, and definability

Lecture 2:

1. Part 1: Index sets
2. Part 2: Applications to classification problems

Computability can be used to measure the complexity of a classification problem.

Computability can be used to prove that there is no classification at all.

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Part 1: Computability and non-computability

Which functions $f : \mathbb{N} \to \mathbb{N}$ are computable?

An early idea (number theory): Study functions which are obtained from simpler functions using some sort of simple rules.

For instance, if we believe that f is computable and g is computable, then $f \circ g$ and $f + g$ (etc.) should also be computable.

Also, we all believe that $f(x) = x + 1$ and $g(x) = 17$ are computable.

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Definition (Herbrand, Gödel 1933)

A function $f : \mathbb{N}^k \to \mathbb{N}$ is recursive if it can be obtained from the basic functions:

- constant functions $C_n^k(x_1, \ldots, x_k) = n$,
- the successor function $S(x) = x + 1$,
- projection functions $P_i^k(x_1, \ldots, x_k) = x_i$,

using finitely many applications of the following operators:

- composition
  $f(x_1, \ldots, x_k) = h(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k))$
- primitive recursion $f(0, x_1, \ldots, x_k) = g(x_1, \ldots, x_k)$
  $f(S(y), x_1, \ldots, x_k) = h(y, f(y, x_1, \ldots, x_k), x_1, \ldots, x_k)$
- minimization

$$\mu(f)(x_1, \ldots, x_k) = z \iff z \text{ is least s.t. } f(z, x_1, \ldots, x_k) = 0.$$

If we omit minimisation we get primitive recursive functions.

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Playing with this definition is similar to writing a basic computer program in some very simple language.

Example

Informally, addition can be recursively defined as follows:

$add(0, x) = x,$

$add(n + 1, x) = add(n, x) + 1.$

Formally,

$add(0, x) = P^1_1(x),$

$add(S(n), x) = S(add(n, x)).$

People (goes back to Kronecker, formally Skolem) in the late 19th -early 20th century checked that all standard number-theoretic functions have recursive definitions.

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Note that when we formally define

$$\mu(f)(x_1,\dots,x_k) = z \iff z \text{ is least s.t. } f(z,x_1,\dots,x_k) = 0,$$

for some arguments $x_1,\dots,x_k$ there could be no such z.

This allows to consider partial recursive functions.

Why is this important?

Theorem

There is no list of all recursive functions $f_0, f_1, \dots$ such that

$$U(i,x) = \text{“the i-th recursive function applied to x”}$$

is itself recursive.

Proof.

Consider $V(i) = U(i,i) + 1 = S(U(i,i))$. If $V = f_j$, then

$$f_j(j) = V(j) = U(j,j) + 1 = f_j(j) + 1.$$

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However, if we allow partial recursive functions then there is a *universal enumeration of all such functions:*

Theorem (Kleene normal form)

There exist primitive recursive T and U such that a partial function f is recursive if and only if there is a number e such that for all n

$$f(n) = U(\mu_x T(e, n, x)).$$

You have the right to ask:

How is this even related to computability?

The answer basically is:

In a way, this universal enumeration is your laptop.

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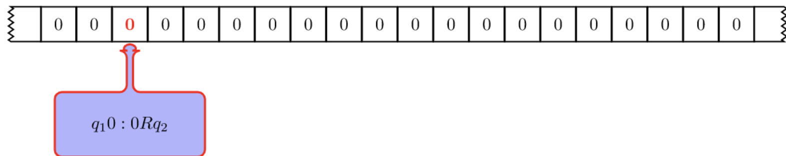
Definition (Turing 1936)

A Turing machine is a mathematical model of computation that defines an abstract machine composed of:

- a potentially infinite tape, and
- a working device with finitely many states



The tape has cells, each cell is either blank or has 0 or 1 in it.
The device can read the symbol in the cell it is currently observing, and based on its current state it can:

1. change the symbol in the cell,
2. move right or left,
3. change the state of the device.

Definition

A function $\mathbb{N} \to \mathbb{N}$ is Turing computable if there is a Turing machine T which, on input the binary representation of x on its tape, finishes its work with the binary representation of $f(x)$ written on its tape.

Theorem (Church, Turing)

A (partial) function $f : \mathbb{N} \to \mathbb{N}$ is recursive if, and only if, it is Turing computable.

Proof idea.

$\to$: By induction. Design a Turing machine for each elementary basic function and explain how to implement composition, primitive recursion, and minimisation.
$\leftarrow$: Design a primitive recursive predicate that says that number $p_0^{y_0} \dots p_n^{y_n}$ codes a valid computation $y_0, \dots, y_n$ of a Turing machine. Then use the minimisation operator to search for a halting computation, and then recover the output. □

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Here are several interesting consequences of this result and Kleene's normal form:

1. All Turing machines can be computably listed:

$$M_0, M_1, M_2, \ldots.$$

2. There exists *the Universal Turing Machine U*:

$$M_e(x) \equiv U(2^e 3^x).$$

The number e is called *the index* of M_e.

By the 1950-s Kleene, Markov, and many others have come up with many other models of computation.

All of these notions eventually were shown to be equivalent to (or even weaker than) Turing computability.

Church-Turing thesis: A function $f : \mathbb{N} \to \mathbb{N}$ can be algorithmically calculated if and only if it is Turing computable.

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For example, let's look at the Halting Problem:

Given a description of a Turing machine, decide whether it halts on its own index.

We need to make it formal.

Definition

A set $X \subseteq \mathbb{N}$ is computable iff its characteristic function

$$\chi(n) = \begin{cases} 1 & \text{if } n \in X \\ 0 & \text{otherwise.} \end{cases}$$

is computable.

One possible formalisation is:

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$$g(e) = \begin{cases} 0 & \text{if } h(e) = 1 \text{ and } M_e(e) = U(2^e 3^e) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

  The function is clearly computable (design a pseudo-code).
- Suppose j is such that $g \equiv M_j$.
- If $g(j) = M_j(j) = 0$ then this means that we are in the case when $h(e) = 1$ and $M_j(j) = U(2^j 3^j) = 1$, which is impossible.
- If $g(j) = M_j(j) = 1$ then we are again in the same case so $g(j) = 0$.

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- Suppose j is such that $g \equiv M_j$.
- If $g(j) = M_j(j) = 0$ then this means that we are in the case when $h(e) = 1$ and $M_j(j) = U(2^j 3^j) = 1$, which is impossible.
- If $g(j) = M_j(j) = 1$ then we are again in the same case so $g(j) = 0$.

Question

Is the set $H = \{e : M_e(e) \text{ halts}\}$ computable?

- Suppose the characteristic function h of H was computable.
- Define

$$g(e) = \begin{cases} 0 & \textit{if } h(e) = 1 \text{ and } M_e(e) = U(2^e 3^e) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

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We have proven:

Theorem

The Halting problem is undecidable.

As long as you can naturally *represent* your problem as a set of natural numbers, it makes sense to ask if the problem is decidable.

A few sample results:

Theorem (Novikov 1955, Boone 1958)

There is a finitely presented group in which the word problem is not computable.

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Part 2: Computability, provability, and definability

§2.1 Hilbert's Tenth Problem. In logic we use **formulas.** There is a **recursive** formal definition of an arbitrary first-order formula which we omit.

Example

Let's restrict ourselves to the ordered semi-ring of natural numbers

$$(\mathbb{N}, +, \times, 0, 1).$$

The **signature** is $\{+, \times, 0, 1\}$. A typical quantifier-free formula $\phi(x, y)$ looks like:

$$((1+1)\times(x\times(x\times y)))+(1+1) = x \ \& \ (x+(1+1+1))\times(y\times x) = y.$$

Assuming the usual axioms it can be re-written as:

$$2x^2y - x + 2 = 0 \ \& \ (x+3)xy - y = 0$$

Evaluating such a formula essentially boils down to evaluating a bunch of polynomial equations.

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Evaluating the corresponding existential projection of $\phi(x, y)$

$$\exists x \exists y \phi(x, y)$$

in $(\mathbb{N}, +, \times, 0, 1)$ is equivalent to evaluating

$$(\exists x)(\exists y)\ [(2x^2y - x + 2)^2 + ((x+3)xy - y)^2 = 0].$$

Thus, it is equivalent to **deciding** whether a polynomial with coefficients in $\mathbb{N}$ (a **Diophantine equation**) has a solution in $\mathbb{N}$.

Problem (Hilbert's Tenth Problem, 1900)

Is there an algorithm which, given a Diophantine equation, decides whether it has a solution (in $\mathbb{N}$)?

We have just argued that if the answer was "yes" then the **existential first-order theory** of $(\mathbb{N}, +, \times, 0, 1)$ would be computable.

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Some preliminary analysis:

Definition

A set is computably enumerable if it is equal to the domain of some (partial) computable function.

Example

Every Diophantine set $\{n : \exists x_1 \ldots \exists x_n p(n, x_0, \ldots, x_n) = 0\}$ is computably enumerable.

Proof.

Define a partial computable function f as follows. List all n-tuples of natural numbers and test whether $p(n, x_0, \ldots, x_n) = 0$ for a given tuple $\bar{x}$. If yes then output 1. □

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The Halting problem $\{e : M_e(e) \text{ halts}\}$ is computably enumerable.

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Theorem (Matiyasevich–Robinson–Davis–Putnam, finished in 1970)

Every computably enumerable set is Diophantine.

Proof.

Hard. □

It follows that the computably enumerable sets are exactly the existentially definable subsets of $\mathbb{N}$!

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Recall that there exists the Universal Turing Machine:

Corollary

There exists a polynomial $u(x, n, y_0, \ldots, y_k)$ with positive integer coefficients such that a set X is computably enumerable (Diophantine) if, and only if, for some $e \in \mathbb{N}$:

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Now we prove the unsolvability of Hilbert's Tenth Problem.

Theorem (Strong unsolvability of Hilbert's Xth Problem)

There is no algorithm which, given e and x, decides whether

$$\exists \bar{y}\,[u(x,e,\bar{y})=0].$$

Proof.

- For a fixed e, define

$$W_e=\{x:\exists \bar{y}\,u(x,e,\bar{y})=0\}=dom\,M_e.$$

- We need to show: $Z=\{2^x3^e : x\in W_e\}$ is not computable.
- The Halting Problem H is computably enumerable but not computable.
- Fix j such that

$$H=W_j.$$

- If Z was computable then so would be its projection on W_j.

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§2.2 Gödel's incompleteness.

Let $H = W_j$, as before.

Theorem (Gödel's incompleteness theorem)

There is a natural number n such that

$$\forall \bar{y} u(n, j, \bar{y}) \neq 0$$

holds but it is not provable in ZFC (the standard set of mathematical axioms).

Proof:

1. ZFC has a computable set of axioms.
2. Every proof is a finite sequence of formulae

$$\phi_0, \dots, \phi_k,$$

where each ϕ_i is either an axiom or is obtained from the previous ϕ_j, $j < i$, using an application of a logical rule (such as modus ponens).

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Proof (continued):

3. This makes the set of provable first-order statements in the arithmetic computably enumerable.
4. In particular, we can computably enumerate provable statements of the form $\forall \bar{y} u(n, j, \bar{y}) \neq 0$.
5. If for all n such that $\forall \bar{y} u(n, j, \bar{y}) \neq 0$ this fact was provable, then it would imply that

$$\bar{W}_j = \bar{H} = \mathbb{N} \setminus H$$

is computably enumerable.
6. To finish the theorem, it is sufficient to recall that H is not computable and also prove:

Lemma (Complementation Theorem)

If both X and $\bar{X}$ are computably enumerable, then X is computable.

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§2.3 The Arithmetical Hierarchy.

Identify sets with their characteristic functions.

Definition

We say that a set Y is computable relative to a set Z, written $Y \leq_T Z$, if there is an oracle Turing machine M_e such that

$$Y \equiv M_e^Z.$$

Think of a hard drive containing Z. Also, think of the keyboard (you are the oracle!).

Here is a stronger version of this:

Definition

$Y \leq_1 Z$ if there is a (total) 1-1 computable f such that

$$x \in Y \iff f(x) \in Z.$$

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Definition

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$$X' = \{e : M_e^X(e) \text{ halts}\}.$$

In particular, $\emptyset' = H = \{e : M_e(e) \text{ halts}\}$, and

$$0^{(n+1)} = (0^{(n)})'.$$

Definition

Define the **arithmetical classes** Σ_n^0 and Π_n^0:

1. $X \in \Sigma_n^0 \iff X \leq_1 0^{(n)}$,
2. $X \in \Pi_n^0 \iff X \leq_1 \overline{0^{(n)}} = \mathbb{N} \setminus 0^{(n)}$,
3. $X \in \Delta_n^0 \iff X \in (\Sigma_n^0 \cap \Pi_n^0)$.

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Theorem

For a set $X \subseteq \mathbb{N}$ and $n > 0$, the following are equivalent:

1. $X \in \Sigma^0_n$;
2. X is computably enumerable relative to $0^{(n-1)}$:

$$\exists e \;\; X = dom\, M_e^{0^{(n-1)}}.$$

3. There is a computable function f of $n+1$ arguments such that

$$X = \{z : \exists x_1 \forall x_2 \ldots Q x_n (f(z, x_1, \ldots, x_n) = 0)\}$$

In fact, f can be replaced with a polynomial over $\mathbb{N}$.

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$$\exists e \;\; X = M_e^{0^{(n-1)}}.$$

3. There exist computable functions f and g of $n+1$ arguments such that

$$X = \{z : \exists x_1 \forall x_2 \ldots Qx_n (f(z, x_1, \ldots, x_n) = 0)\}$$

and

$$X = \{z : \forall x_1 \exists x_2 \ldots Rx_n (g(z, x_1, \ldots, x_n) = 0)\}.$$

Theorem

For a set $X \subseteq \mathbb{N}$ and $n > 0$, the following are equivalent:

1. $X \in \Delta_n^0$;
2. X is computable relative to $0^{(n-1)}$:

$$\exists e \ X = M_e^{0^{(n-1)}}.$$

3. There exist computable functions f and g of $n+1$ arguments such that

$$X = \{z : \exists x_1 \forall x_2 \ldots Q x_n (f(z, x_1, \ldots, x_n) = 0)\}$$

and

$$X = \{z : \forall x_1 \exists x_2 \ldots R x_n (g(z, x_1, \ldots, x_n) = 0)\}.$$

The famous diagram (in the Soare's book)