Pascal's triangle, Padé approximations and an application

The following triangular array is formed by adding adjacent cells in a row, to give the cell between them on the next row. This is just as for Pascal's triangle, even though the cells are vectors of polynomials rather than integers.

$$\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1-z\end{bmatrix} \qquad \begin{bmatrix}1+z\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1-z+\frac{1}{2}z^2\end{bmatrix} \qquad \begin{bmatrix}2+z\\2-z\end{bmatrix} \qquad \begin{bmatrix}1+z+\frac{1}{2}z^2\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1-z+\frac{1}{2}z^2-\frac{1}{6}z^3\end{bmatrix} \qquad \begin{bmatrix}3+z\\3-2z+\frac{1}{2}z^2\end{bmatrix} \qquad \begin{bmatrix}3+2z+\frac{1}{2}z^2\\3-z\end{bmatrix} \qquad \begin{bmatrix}1+z+\frac{1}{2}z^2+\frac{1}{6}z^3\\1\end{bmatrix}$$

The vectors on row number $k = 0, 1, 2, \ldots$ represent approximations to $\exp(z)$ of order k in the sense that the rational function formed from the two components, say $N(z)$ and $D(z)$, of any cell on this row satisfies

$$\frac{N(z)}{D(z)} = \exp(z) + Cz^{k+1} + O(z^{k+2}).$$

The values of C corresponding to row number k have the same magnitude but alternate in sign. This is why adding an adjacent pair to form an entry in the next row of the triangle increases the order by 1.

Many relationships exist between triples of entries in this table and we will explore one simple example. The relationship is between the central three entries on rows $2n-4$, $2n-2$ and $2n$. Denote the entry in the centre of row $2n$ by

$$V_n(z) = \begin{bmatrix}N_n(z)\\D_n(z)\end{bmatrix}$$

and we have

$$V_n(z) = \alpha_n V_{n-1}(z) + \beta_n z^2 V_{n-2}(z), \qquad n = 2, 3, \ldots, \tag{1}$$

where $\alpha_n = 2(2n-1)/n$, $\beta_n = 1/n(n-1)$. The right-hand side represents an approximation to $\exp(z)$ with order at least $2n-2$, irrespective of the values of $\alpha_n \neq 0$ and β_n. One can easily verify that the values actually used, give the correct z^0 and z^n terms in $V_n(z)$ and a proof of (**??**) can be built on these observations.

Given a function f, assumed to be analytic in a neighbourhood of zero with $f(0) \neq 0$, there may exist for particular non-negative integers l and m a pair of polynomials N of degree l, and D of degree m, such that $N(z)/D(z) = f(z) + O(z^{l+m+1})$. In this case the rational function N/D is known as an (l,m)-Padé approximation to f. For some functions the complete Padé table exists and, as we see by rotating the Pascal's triangle we have introduced into tabular form, exp is one of these functions. The tabular arrangement in this case, where a few more entries have been squeezed in, is

	$l=0$	$l=1$	$l=2$	$l=3$
$m=0$	$\frac{1}{1}$	$\frac{1+z}{1}$	$\frac{1+z+\frac{1}{2}z^2}{1}$	$\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3}{1}$
$m=1$	$\frac{1}{1-z}$	$\frac{2+z}{2-z}$	$\frac{3+2z+\frac{1}{2}z^2}{3-z}$	$\frac{4+3z+z^2+\frac{1}{6}z^3}{4-z}$
$m=2$	$\frac{1}{1-z+\frac{1}{2}z^2}$	$\frac{3+z}{3-2z+\frac{1}{2}z^2}$	$\frac{6+3z+\frac{1}{2}z^2}{6-3z+\frac{1}{2}z^2}$	$\frac{10+6z+\frac{3}{2}z^2+\frac{1}{6}z^3}{10-4z+\frac{1}{2}z^2}$
$m=3$	$\frac{1}{1-z+\frac{1}{2}z^2-\frac{1}{6}z^3}$	$\frac{4+z}{4-3z+z^2-\frac{1}{6}z^3}$	$\frac{10+4z+\frac{1}{2}z^2}{10-6z+\frac{3}{2}z^2-\frac{1}{6}z^3}$	$\frac{20+10z+2z^2+\frac{1}{6}z^3}{20-10z+2z^2-\frac{1}{6}z^3}$

Now the application. There exists an ordinary differential equation counterpart to Gauss-Legendre quadrature in which, for each time step of length Δt, the solution is advanced using an implicit Runge-Kutta method, containing n stages evaluated at the zeros of the Legendre polynomial P_n, adapted to the interval $[t, t+\Delta t]$. The error generated in a step is equal to $O(\Delta t^{2n+1})$. To be acceptable for the solution of the type of "stiff" problems arising in the discretisation of time-dependent partial differential equations, the numerical method must be stable for the solution of $y' = \lambda y$ whenever λ is in the left half-complex-plane. Write $z = \lambda \Delta t$ and this means that the value of the (n,n) Padé approximation to $\exp(z)$ must lie in the unit disc, whenever z is in the left half-plane. Using (**??**) it follows that this approximation can be written in continued fraction form as

$$R(z) = 1 + \frac{1}{-\frac{1}{2}+\frac{1}{z}+}\,\frac{1}{\frac{12}{z}+}\,\frac{1}{\frac{5}{z}+}\,\cdots\,\frac{1}{\frac{a_n}{z}},$$

where $a_n = 4(2n-1)$ for n even, and $a_n = 2n-1$ for n odd. The proof that $|R(z)| < 1$ for Re $z < 0$, is left as an exercise, but hinges on the facts that the left half-plane is closed under addition, multiplication by a positive real and by the mapping $z \mapsto z^{-1}$.

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