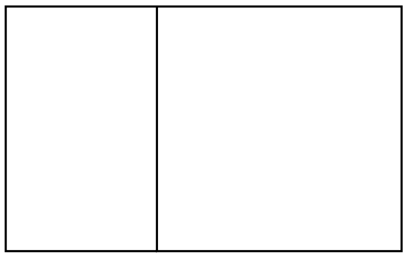
Mathematical Apology 18: Golden Days

The fascination with the "Golden Ratio", dating from ancient times does not seem to have abated. In the penultimate meeting of a first year University class I asked for suggestions for filling in the remaining few minutes of the hour. I expected a plea for the revision of difficult topics but instead I was asked to talk about the Golden Ratio. The three minute impromptu lecture which resulted, is here expanded to about six minutes.

In the diagram



the ratio of the height to the width of the rectangle to the left of the square is the same as the ratio of the width to the height of the large rectangle making up the whole figure. Call this ration ϕ and we see that

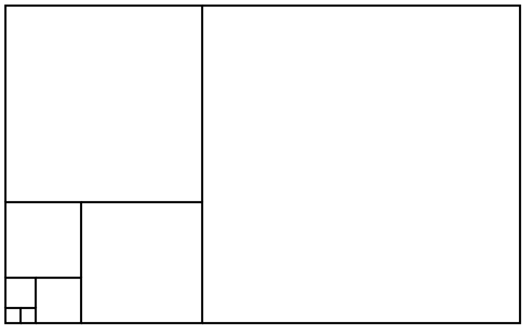
$$\phi : 1 = 1 + \phi : \phi.$$

This means that $\phi^2 = \phi + 1$ and therefore (because $\phi > 0$) that

$$\phi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618033989.$$

This is the famous Golden Ratio.

We can start with a small rectangle and keep building it up in size by adding a square to the longer of its two sides (or to either side if we start with a square). We get something like this



The ratios of the longer to the shorter sides of the sequence of rectangles are calculated in the following table:

long side	short side	ratio
1	1	1.000000
2	1	2.000000
3	2	1.500000
5	3	1.666667
8	5	1.600000
13	8	1.625000
21	13	1.615384
34	21	1.619048

We recognise the integers appearing in this table as Fibonacci numbers and we observe that the ratios seem to be getting closer and closer to ϕ.

Hence, in thinking about the Golden Ratio we need to think also about Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}, \qquad n = 2, 3, 4, \ldots.$$

Leaving aside F_0, these are exactly the numbers which occur as lengths of the sides of the rectangles in the second diagram.

A difference equation of the form

$$u_n + au_{n-1} + bu_{n-2} = 0$$

has general solution

$$u_n = A\lambda^n + B\mu^n$$

where λ and μ are solutions to the polynomial equation

$$x^2 + ax + b = 0,$$

assuming that this quadratic equation has distinct roots. If u_0 and u_1 are given, to start the sequence off, then the "arbitrary constants" A and B have specific values defined by

$$A + B = u_0, \qquad A\lambda + B\mu = u_1.$$

In the case of the Fibonacci numbers,

$$a = b = -1, \qquad \lambda = \phi, \quad \mu = 1 - \phi = -\phi^{-1}, \qquad A = -B = \frac{1}{\sqrt{5}}.$$

Hence, we have a formula for the Fibonacci number F_n and for the ratio F_n/F_{n-1}:

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi)^{-n}), \qquad \frac{F_n}{F_{n-1}} = \frac{\phi^n - (-\phi)^{-n}}{\phi^{n-1} - (-\phi)^{-n+1}}.$$

To verify the limiting value of the ratio F_n/F_{n-1}, write it in the form

$$\frac{F_n}{F_{n-1}} = \left(\frac{1 + (-1)^{n+1}\phi^{-2n}}{1 + (-1)^n\phi^{-2n+2}}\right)\phi,$$

which is ϕ multiplied by a factor whose limiting value is 1.

Given a positive real number, ξ, we can write it as a continued fraction

$$\xi = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

The values of a_0, a_1, a_2, ... are easily found by defining a_0 as the integer part of ξ, a_1 as the integer part of $(\xi - a_0)^{-1}$, a_2 as the integer part of $((\xi - a_0)^{-1} - a_1)^{-1}$ and so on. The approximations formed by truncating the sequence after various numbers of terms are known as convergents and give surprisingly good approximations to ξ.

In the special case that $\xi = \phi$, it turns out that the continued fraction is just

$$\phi = [1, 1, 1, 1, 1, \ldots]$$

and convergent number n is F_{n+1}/F_n.

Interest in Fibonacci numbers and the Golden Ratio is increasing at a rate determined by how long it takes an enthusiast to explain their wonderful properties to someone else and to convert this person into a similarly committed enthusiast. Unfortunately, this level of enthusiasm only lasts for two time steps. Suppose that after n units of this basic time interval have elapsed, the number of new recruits is u_n. The value of u_n satisfies the difference equation

$$u_n = u_{n-1} + u_{n-2},$$

because only the people recruited in the previous two time steps are themselves recruiting. It follows that u_n increases by a factor ϕ per time step.

You can find many web-sites by searching for "Golden ratio" and amongst the information readily available is that there is Mathematical periodical known as *The Fibonacci Quarterly* which publishes only articles concerned with these matters.

If there are Golden Days in a person's life then these are, for many people, the days when they are students and have time to look at many special and beautiful branches of Mathematics, just because they are there. Fibonacci Numbers and the Golden Ratio are accessible to anyone and have much to commend themselves in terms of the sheer joy they can bring to almost anybody.