

Mathematical Apology 17: Mathematical Connections

J. C. Butcher

As a retired member of a University Mathematics Department, there is little I can do to influence the way my subject develops, except to interfere whenever the opportunity arises. Such an opportunity came my way a few days ago, when a new student, who did not know where to go or whom to speak to, crossed my threshold by chance. He had a firm view that he wanted to study Pure Mathematics, because he didn't like the vagueness and lack of rigour that he associated with Applied Mathematics. Of course the closest he had ever got to studying Applied Mathematics at High School had been in the context of "Mathematics with Statistics" and all his prejudices had stemmed from this experience. I seemed to have a window of opportunity lasting just a few minutes. In the time it took me to find out to whom he should be speaking, and to take him to that person, I wanted to correct what I believe were his wrong beliefs or at least to put some doubts into his head. I don't think I succeeded.

Some people think that Mathematics should be taught as an essentially intellectual subject, in which very little is left to conjecture or intuition. Another view is that Mathematics contains a large number of facts, many of which are useful; learning mathematics is learning these facts and learning how to use them. A middle view is that Mathematics is both logical and useful and that studying Mathematics is learning about the richness of the overall structures, both from the point of view of their meanings and interconnections, as well as from the roles they might play in science and other subjects, where Mathematics has been found to have applications.

A political leader said recently that the old divisions between Left and Right were now out of date but, rather sanctimoniously, that he did know the difference between Right and Wrong. I think that the old divisions between Pure and Applied Mathematics are also out of date but that we should

foster Good Mathematics. Like the politician's truism, this doesn't have any absolute meaning but represents an ideal to aim for.

It always excites me when I realise that some apparently unconnected ideas in Mathematics actually do have a connection. Here is an instance of this type of good fortune.

Problem 1 Let t denote a real number such that $t \notin \{0, 1\}$. Find the integral

$$\int \frac{dx}{(x-t)x^2(x-1)^2}$$

Problem 2 Find a_0, b_0, a_1, b_1 such that, the interpolation formula

$$f(t) \approx a_0 f(0) + b_0 f'(0) + a_1 f(1) + b_1 f'(1). \quad (1)$$

is exact whenever f is a polynomial with degree not exceeding 3,

The first step in solving Problem 1 is to find the partial fraction form of the integrand

$$\frac{1}{(x-t)x^2(x-1)^2} = \frac{1}{t^2(t-1)^2} \frac{1}{x-t} - \frac{2t+1}{t^2} \frac{1}{x} - \frac{1}{t} \frac{1}{x^2} + \frac{2t-3}{(t-1)^2} \frac{1}{x-1} - \frac{1}{(t-1)} \frac{1}{(x-1)^2} \quad (2)$$

and the rest is easy.

The straightforward, but unsophisticated, way of solving Problem 2 is to obtain linear equations satisfied by a_0, b_0, a_1, b_1 , by substituting $f(x) = 1, x, x^2, x^3$ into (1). The solution of these equations is

$$a_0 = (t-1)^2(2t+1), \quad a_1 = t^2(3-2t), \quad b_0 = t(1-t)^2, \quad b_1 = t^2(t-1).$$

Before calculators and computers became generally available, special functions like exp had to be found from tables, and interpolation became necessary to keep the tables down to a reasonable size. The solution to Problem 2 is known as an Hermite interpolation formula and gives an approximation to $f(t)$ from known values and derivatives at $x = 0$ and $x = 1$. For example, to calculate $\exp(0.05)$ from known values $\exp(0) = 1$ and $\exp(0.1) = 1.105171$, use (1) with $f(x) = \exp(x/10)$. The interpolated value is 1.051271, which is exact to 6 decimal places.

What is the connection between the two problems? Go through the terms on the right hand side of (2), and replace terms like $(x-\xi)^{-1}$ by $f(\xi)$, for

$\xi = t, 0, 1$, and terms like $(x - \xi)^{-2}$ by $f'(\xi)$. Equate this transformed right hand side to zero, rearrange, and the interpolation formula is found.

I regard interpolation questions as Applied Mathematics, because computational schemes for numerical integration and differential equations hinge on them. The elegant proof of the relationship between questions like Problem 1 and Problem 2 uses calculus in the complex plane, which is regarded as Pure Mathematics. I really hope that the student I would like have influenced studies the whole gamut of Analysis and Algebra, which are at the heart of Mathematics. But I hope that he also finds his way into those parts of the Mathematical Sciences which explore deep and beautiful scientific problems.