Department of Mathematics

Maths 761 Assignment 3

September 21, 2010 Due: 4pm, Thursday 30th September, 2010

This assignment is worth 10% of the final grade for this course. Hand in this assignment to the box outside SciSpace on the ground floor of the Maths Building. Show all your working.

In this assignment, you can use XPP whenever it is useful, but be careful how you present any answers that depend on numerical evidence: support statements based on numerical evidence with appropriate theoretical calculations and include limited appropriate computer printouts with your answer.

1. Consider the system

$$\begin{aligned}\dot{x} &= x(1-y),\\ \dot{y} &= -y + x^2.\end{aligned}$$

Draw a careful phase portrait for this system, making sure to include the dynamics near to any equilibria, and including the stable and unstable manifolds of any saddle points. Make sure you justify the existence, or lack of, any periodic orbits. You may find it helpful to consider any symmetries or invariant axes of the system. *Note: You do not need to calculate power series expansions for the stable and unstable manifolds.*

2. Consider the system:

$$\begin{aligned}\dot{x} &= (1+\mu)x - 4y + x^2 - 2xy,\\ \dot{y} &= 2x - 4\mu y - y^2 - x^2,\end{aligned}$$

(a) Show that the origin $(x, y) = (0, 0)$ is a non-hyperbolic equilibrium when $\mu = -2$.

(b) Use centre manifold theory to work out the dynamics near the origin for $\mu \approx -2$. (Hint: you will need to define an extended system in which μ appears as a variable, then find a power series expansion for the extended centre manifold, then compute an equation for the dynamics on that extended centre manifold. Lastly, analyze the dynamics associated with that equation.) Draw a bifurcation diagram for the system valid for $\mu \approx -2$.

(c) Sketch phase portraits showing the behaviour of solutions near the origin for $\mu \approx -2$.

Note that XPP computations are frequently unreliable near non-hyperbolic equilibria.

3. The following system is from Lecture 15 and Lab 8.

$$\begin{aligned}\dot{x} &= x(\mu + x^2 - 0.5y^2),\\ \dot{y} &= y(4 + x^2 - y^2).\end{aligned}$$

(a) Find and discuss all bifurcations in this system. Note that this part of the question asks you to repeat calculations done in class. You should special attention to the way you present your results – your presentation should be sufficiently clear that it could be understood by a 4th year Maths student who is not enrolled in Maths 761.

(b) Use XPPAUT to compute a bifurcation diagram for this system. Make sure that your numerical results match the analytical results obtained in (a) and that your bifurcation diagram is fully labelled and explained.

4. Consider the differential equation

$$\dot{x} = \mu - x + x^3.$$

(a) Use XPPAUT to plot a bifurcation diagram for this equation.

(b) Use the bifurcation theorems given in class to identify any bifurcations that appear in your bifurcation diagram. (Note that the shape of the bifurcation diagram will give you a very good idea of what the bifurcations are but you must check the conditions of the theorems and present your result clearly to get full marks for this question.)

5. (Harder) Discuss the bifurcations seen in the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu + y - x^2 - xy \end{aligned}$$

where μ is a parameter. A complete answer to this question will include a bifurcation diagram and phase portraits as well as a discussion of any parts of the dynamics that you are uncertain about. Include a limited amount of XPP output if it is helpful to your explanation.