DEPARTMENT OF MATHEMATICS

MATHS 761 - Worked examples of bifurcations in systems with two parameters

Example 1 (cusp bifurcation)

Consider the differential equation

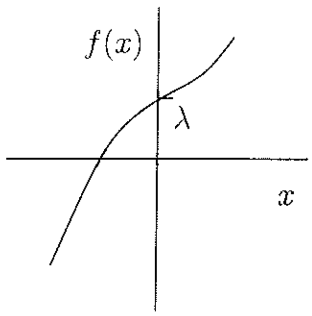
$$\dot{x} = \lambda + \mu x + x^3$$

Write $f(x) = \lambda + \mu x + x^3$. We wish to find roots of f, which correspond to stationary solutions of the differential equation.

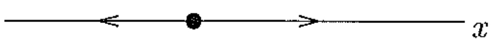
Turning points of f are values of $\dot{x}$ satisfying

$$f'(x) = \mu + 3x^2 = 0$$

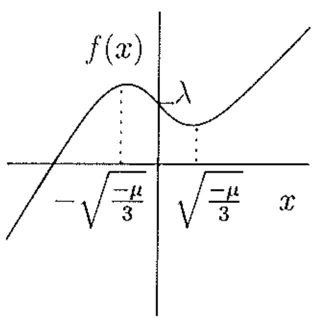
So f has turning points only if $\mu < 0$, in which case the turning points are $x = \pm\sqrt{-\mu/3}$. Hence, if $\mu > 0$ the graph of f looks like:



Thus f has a single root, with the root being negative if $\lambda > 0$ and positive if $\lambda < 0$. Also, $Df = \mu + 3x^2 > 0$ for $\mu > 0$ so the single stationary solution is unstable. The phase portrait will be qualitatively the same for all λ when $\mu < 0$, i.e.,



If $\mu < 0$, the graph of f looks like:



The value of f at the turning points is given by:

$$
\begin{aligned}
f\left(\sqrt{\frac{-\mu}{3}}\right) &= \lambda - \frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu \\
f\left(-\sqrt{\frac{-\mu}{3}}\right) &= \lambda + \frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu
\end{aligned}
$$

Thus, there will be:

- one real, negative root of f if $\lambda - \frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu > 0$;
- three real roots if $\lambda - \frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu < 0 < \lambda + \frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu$;
- one real, positive root if $\lambda + \frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu < 0$.

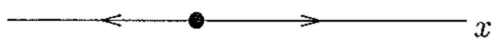
Bifurcations occur at $\lambda = \pm\frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu$, and we see that these bifurcations are probably saddle-node bifurcations.

To compute stability of the equilibria, note that $Df = \mu + 3x^2$ and this is negative if

$$-\sqrt{\frac{-\mu}{3}} < x < \sqrt{\frac{-\mu}{3}}.$$

We conclude that if there is only one stationary solution it is unstable while if there are three, the one with the middle value of x is stable and the other two are unstable.

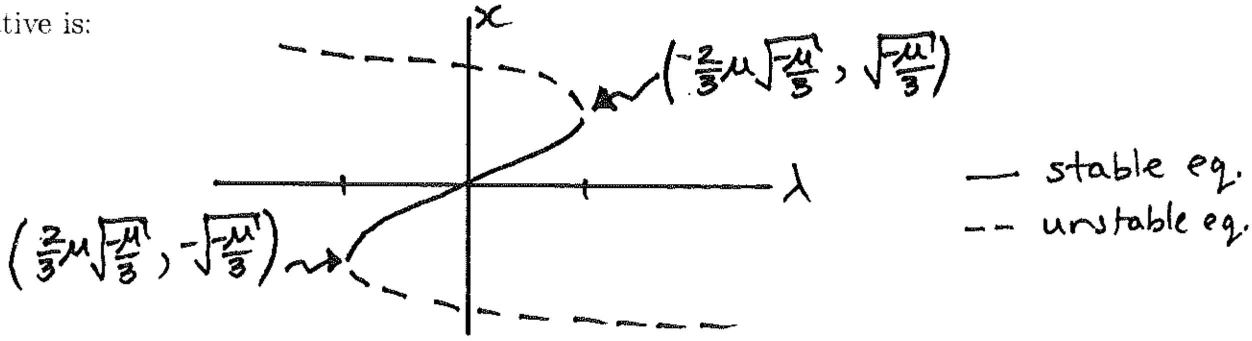
The phase portraits are as follows.
For $\lambda > \frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu$ (one root of f, with $x < -\sqrt{\frac{-\mu}{3}}$):



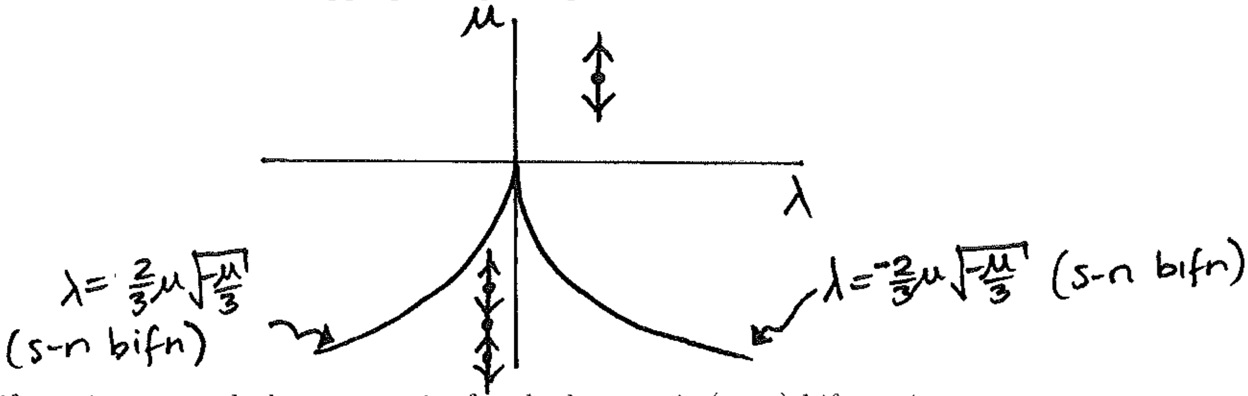
For $-\frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu < \lambda < \frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu$ (3 roots of f):

For $\lambda < -\frac{2}{3}\sqrt{\frac{-\mu}{3}}\mu$ (one root of f, with $x > \sqrt{\frac{-\mu}{3}}$):

A bifurcation diagram showing what happens if λ is varied while μ is held constant and negative is:



We can represent all this information (for $\mu < 0$ and $\mu > 0$) by drawing a *bifurcation set*, which marks the places in the (λ, μ)-plan where bifurcations occur. In each region of the plane we then draw the appropriate phase portrait. We find



Bifurcation set and phase portraits for the hysteresis (cusp) bifurcation.

Example 2 (Bogdanov-Takens bifurcation)

Consider the system

$$\begin{aligned}\dot{x} &= \lambda - \mu x + y^2 + xy \\ \dot{y} &= x\end{aligned}$$

Stationary solutions occur at $x = 0, y = \pm\sqrt{-\lambda}$ if $\lambda < 0$ (and so we suspect there is a saddle-node bifurcation at $\lambda = 0$).

To compute stability, note that

$$Df(0, \sqrt{-\lambda}) = \begin{pmatrix} -\mu + \sqrt{-\lambda} & 2\sqrt{-\lambda} \\ 1 & 0 \end{pmatrix}$$

which has eigenvalues t satisfying $t^2 + (\mu - \sqrt{-\lambda})t - 2\sqrt{-\lambda} = 0$.

Since $\det(Df(0, \sqrt{-\lambda})) < 0$, this stationary solution is a saddle. Also

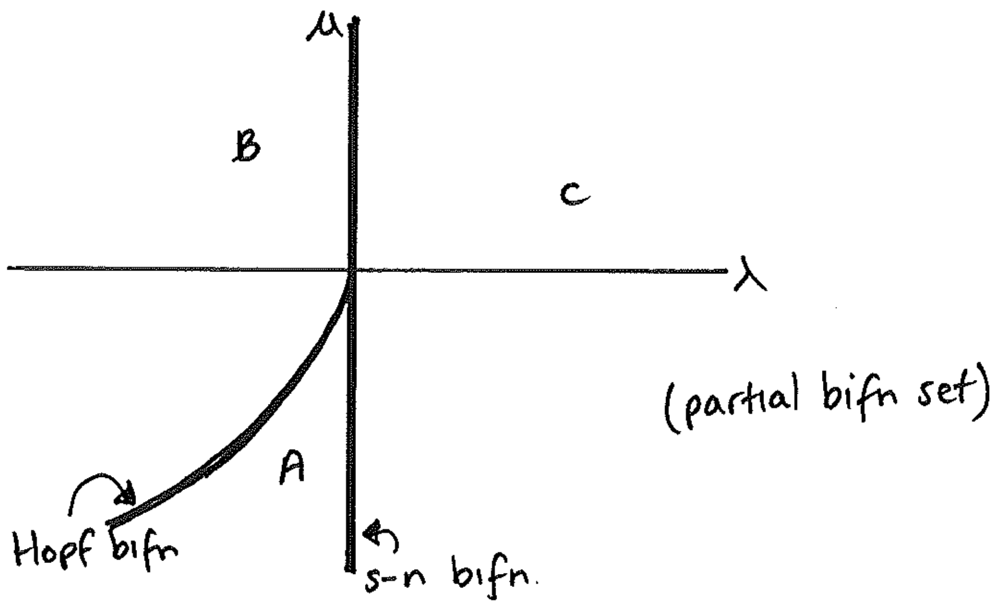
$$Df(0, -\sqrt{-\lambda}) = \begin{pmatrix} -\mu - \sqrt{-\lambda} & -2\sqrt{-\lambda} \\ 1 & 0 \end{pmatrix}$$

which has positive determinant and trace$= -\mu - \sqrt{-\lambda}$. We conclude that this stationary solution changes from a source to a sink at (λ, μ) such that $\mu + \sqrt{-\lambda} = 0$ i.e., when $\lambda = -\mu^2$ and $\mu \leq 0$.

At $\lambda = -\mu^2$, the determinant of $Df(0, -\sqrt{-\lambda})$ is $2\sqrt{-\lambda} > 0 \Rightarrow$ the eigenvalues of the Jacobian are $\pm\sqrt{2}i\sqrt{\sqrt{-\lambda}}$, i.e., there is a Hopf bifurcation at $\lambda = -\mu^2, \mu \leq 0$.

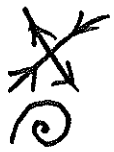
To work out whether $(0, -\sqrt{-\lambda})$ is a source or a sink near the saddle-node bifurcation, note that for $\lambda \approx 0$, the eigenvalues of $Df(0, -\sqrt{-\lambda})$ satisfy $t^2 + \mu t \approx 0$ or $t \approx 0, -\mu$. We conclude that $(0, -\sqrt{-\lambda})$ is a sink near $\lambda = 0$ if $\mu > 0$ and a source if $\mu < 0$.

Putting all the information we have so far onto a bifurcation set, we find



In region A, close to $\lambda = 0$, the phase portrait is (qualitatively)

In region B, close to $\lambda = 0$, the phase portrait is (qualitatively)



On which side of the Hopf bifurcation does the periodic orbit exist? What happens to this periodic orbit away from the Hopf bifurcation and how will this modify the bifurcation set?