DEPARTMENT OF MATHEMATICS
MATHS 761 Worksheet 5 - Extended centre manifolds

This worksheet guides you through the procedure for calculating the dynamics near the point $(x, y) = (0, 0)$ when $\mu \approx 1$, for the system of equations:

$$\begin{aligned} \dot{x} &= y - x - x^2, \\ \dot{y} &= \mu x - y - y^2. \end{aligned}$$

At the end of this exercise you will be able to describe the dynamics that occurs on the extended centre manifold and use this to describe the behaviour of solutions near the bifurcation at $\mu = 1$.

Step 1 : Show that $x = y = 0$ is a nonhyperbolic stationary point for the system above when $\mu = 1$.

Step 2 : Consider the case $\mu = 1$. Show that the change of coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

puts the linearised system into diagonal form.

Step 3 : Define a new parameter $\lambda = \mu - 1$. Derive the differential equations for the new coordinates and write down the new extended system

$$\begin{aligned} \dot{u} &= \ldots, \\ \dot{v} &= \ldots, \\ \dot{\lambda} &= 0. \end{aligned}$$

Step 4 : Show that the centre manifold for the extended system calculated in Step 3 has an approximate Taylor series expansion $v = -1/4u\lambda$.

To do this, note first that in the extended system, λ is regarded as a variable, which means that terms such as λu are *quadratic* terms. Using this observation, you should be able to see that the centre manifold in the extended system is two-dimensional and is tangent to the $\lambda - u$ plane at the point $(u, v, \lambda) = (0, 0, 0)$. Thus, the extended centre manifold can be described by the equation $v = h(\lambda, u)$ for some function h.

We look for a power series expansion of h, i.e. write $v = a + b\lambda + cu + d\lambda^2 + e\lambda u + fu^2 + \ldots$. Now $a = b = c = 0$ because the extended centre manifold is tangent to the $u - \lambda$ plane at $(u, v, \lambda) = (0, 0, 0)$. As usual, we calculate two expressions for $\dot{v}$ and equate terms to find the values of d, e and f. The first expression comes from

$$\dot{v} = \frac{\partial h}{\partial \lambda}\dot{\lambda} + \frac{\partial h}{\partial u}\dot{u} = \frac{\partial h}{\partial u}\dot{u}$$

since $\dot{\lambda} = 0$. The second expression for $\dot{v}$ comes from the differential equation calculated in Step 3. Equating terms in the two expressions yields $d = f = 0$, $e = -1/4$.

Step 5 : Show that on the extended centre manifold the dynamics obeys the equations

$$\begin{aligned}\dot{u} &= 0.5u(\lambda - 2u) + \text{higher order terms},\\ \dot{\lambda} &= 0.\end{aligned}$$

Step 6 : Determine the behaviour of solutions on the extended centre manifold. Do this by analysing the position and stability of stationary solutions for the system derived in Step 5. Equivalently, think of λ as a constant again and determine the position and stability of stationary solutions to the equation $\dot{u} = 0.5u(\lambda - 2u)$ for various values of λ near zero.

Draw phase portraits for various values of λ and draw an appropriate bifurcation diagram (i.e., plot the u coordinates of all stationary solutions as a function of λ). Indicate the stability of the stationary solutions by using different line styles to represent the different stabilities.

Step 7 : Describe the behaviour of solutions to the original differential equations near $x = y = 0$ for μ near 1.

Step 8 : Why is your answer to Step 7 valid only for x and y near zero and for μ near 1?