Maths 761 Lecture 16

- **Topic for today:** The Hopf bifurcation
- **Reading for this lecture:** Glendinning §8.8, to end of Example 8.8
- **Suggested exercises:** Worksheet 6
- **Reading for next lecture:** Glendinning §8.8
- **Today's handout:**
  Lecture 16 summary
  Worksheet 6

Hopf bifurcations

We consider the bifurcation of a stationary solution that occurs when the Jacobian has one pair of purely imaginary eigenvalues.

In this case the centre manifold is two-dimensional and we need to consider the possibility that periodic orbits may occur near the bifurcation.

Example 1:

Determine the dynamics in the following system for $\mu \approx 0$.

$$\begin{aligned}\dot{x} &= -\mu x + 0.5y + x(x^2 + y^2), \\ \dot{y} &= -0.5x - \mu y + y(x^2 + y^2).\end{aligned}$$

The Hopf bifurcation theorem

Consider the system

$$\dot{x} = f(x, y; \mu), \;\; \dot{y} = g(x, y; \mu),$$

where $x, y, \mu \in \mathbf{R}$, $f(0, 0; \mu) = g(0, 0; \mu) = 0$, and the Jacobian evaluated at the origin when $\mu = 0$ is

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

for some constant $\omega \neq 0$. Define

$$\begin{aligned} a = & \frac{1}{16}(f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy}) \\ & + \frac{1}{16\omega}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}.\right] \end{aligned}$$

Then if $f_{\mu x} + g_{\mu y} \neq 0$ and $a \neq 0$, a curve of periodic solutions bifurcates from the origin into $\mu > 0$ if $a(f_{\mu x} + g_{\mu y}) < 0$ or into $\mu < 0$ if $a(f_{\mu x} + g_{\mu y}) > 0$.

Hopf bifurcation theorem continued...

The origin is stable for $\mu < 0$ and unstable for $\mu > 0$ if $f_{\mu x} + g_{\mu y} > 0$, with stabilities reversed if $f_{\mu x} + g_{\mu y} < 0$. The periodic solutions are stable if the origin is unstable on the side of $\mu = 0$ that the periodic solutions exist, and vice versa.

The amplitude of the periodic orbit grows as $\sqrt{|\mu|}$ for $|\mu|$ near zero, and the period of the orbit tends to $2\pi/\omega$ as $|\mu|$ tends to zero.

- A Hopf bifurcation is *supercritical* if on the centre manifold the bifurcating periodic orbits are stable. Otherwise the bifurcation is said to be *subcritical*.
- The theorem assumes there is a fixed point at $(x, y) = (0, 0)$ for all values of μ. We may have to change coordinates to translate the fixed point to the origin before applying the theorem.

Example 2

Check that the conditions of the Hopf bifurcation theorem hold at $\mu = 0$ in Example 1:

$$\begin{aligned} \dot{x} &= -\mu x + 0.5y + x(x^2 + y^2), \\ \dot{y} &= -0.5x - \mu y + y(x^2 + y^2). \end{aligned}$$

Example 3

Find the bifurcations that occur as λ is varied in the system

$$\begin{aligned}\dot{x} &= \lambda x + 2xy + xy^2, \\ \dot{y} &= 1 - x^2 - y^2.\end{aligned}$$