DEPARTMENT OF MATHEMATICS

Maths 761 Assignment 4

October 6, 2010 Due: 4pm, Thursday 14th October, 2010

This assignment is worth 10% of the final grade for this course. Hand in this assignment to the box outside SciSpace on the ground floor of the Maths Building. Show all your working.

In this assignment, you can use XPP whenever it is useful, but be careful how you present any answers that depend on numerical evidence: support statements based on numerical evidence with appropriate theoretical calculations and include limited appropriate computer printouts with your answer.

1. This question is about local bifurcations in the following systems of equations.

(a)

$$\begin{aligned} \dot{x} &= 4y + x(\mu - 2 + y^2), \\ \dot{y} &= -x. \end{aligned}$$

(b)

$$\begin{aligned} \dot{x} &= x - y, \\ \dot{y} &= (y+1)(x^2 - \mu). \end{aligned}$$

In each case

(i) find all values of μ for which there is a bifurcation,

(ii) draw the bifurcation diagram (by hand), and

(iii) identify any bifurcations that you find.

Note: You may classify any bifurcations you find from the bifurcation diagram. You do not need to perform any centre manifold calculations or check the conditions from any bifurcation theorems.

2. This question is concerned with the following system of equations.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \lambda + \mu y + x^2 + xy. \end{aligned}$$

(a) Find the local bifurcations in the system analytically.

(b) Use XPP to locate the position of the homoclinic bifurcation curve in the $\lambda - \mu$ plane.

(c) Find a path in the $\lambda - \mu$ plane that cuts both the Hopf bifurcation curve and the homoclinic bifurcation curve. Use XPP to get the bifurcation diagram along this path, including following the periodic orbit created in the Hopf bifurcation until it has high period (say a period of about 200).

(d) For the branch of periodic orbits you followed in (c), plot a graph of the period of the periodic orbit vs. the natural log of $|\lambda - \lambda^*|$, where λ^* is the value of λ at which the homoclinic bifurcation occurs. Compare the slope of this line with the prediction from the theory we developed in class.

3. (Harder) Construct a geometric return map to investigate the existence and stability of periodic orbits near a planar heteroclinic bifurcation with the following characteristics, as illustrated in the figure below.

- There are two equilibria of saddle type for all values of μ, the bifurcation constant. Call the equilibria A and B, as in the picture.
- There is a heteroclinic connection from B to A for all values of μ.
- If $\mu = 0$, there is a heteroclinic connection from A to B.
- If $\mu \neq 0$, the stable and unstable manifolds of A and B have the orientation shown in the figure.

Use your map to determine conditions on the eigenvalues of the linearised flow near each equilibrium such that the periodic orbit involved in the heteroclinic bifurcation is stable. Clearly state the assumptions you make in constructing the return map. Say whether the eigenvalue condition you get in the end seems to fit with the theory for planar homoclinic bifurcations.

Hints on how to proceed:

(a) Draw some schematic phase portraits that show that, generically, a periodic orbit is destroyed or created in a heteroclinic bifurcation of this type.

(b) Put small boxes around each of the equilibria. Define two sets of local coordinates, one is each of the boxes. Define local maps from one side of each box to another side. Use the linearised flow to get an approximation to each local map.

(c) Define global maps from the outgoing side of the box around A to the ingoing side of the box about B, and from the outgoing side of the box around B to the ingoing side of the box about A. Use Taylor expansions to approximate each global map. Note that the map from B to A will not depend on μ, to lowest order.

(d) Take a composition of the maps to get a return map. Look for fixed points of the composite map.