Maths 761 Assignment 1

July 20, 2010 Due: 4pm, Tuesday August 3rd

This assignment is worth 10% of the final grade for this course. Hand in this assignment to your lecturer on the correct day. You are encouraged to use XPP to check your answers, but you must make sure to carefully justify analytically any statements you make, and show all your working. Starred (*) questions are trickier than the rest, and should not be attempted until you have completed all the other questions.

1. (15 marks) Consider the following system of differential equations:

$$\dot{x} = (1-y)(x-2),$$
$$\dot{y} = x^2 - y.$$

   (a) Find all equilibrium solutions and determine their types. For each equilibrium you find, sketch a phase portrait showing the behaviour of solutions near that equilibrium.

   (b) Carefully draw the global phase portrait. Use nullclines to help draw the picture accurately.

   (c) Use XPP to confirm your answers, and hand in a phase portrait plotted with XPP with your answers.

2. (10 marks) Consider the equations:

$$\dot{x} = x - y + x^2 - 2xy,$$
$$\dot{y} = -y + x^2.$$

   (a) Find the power series expansions for $W^u_{\text{loc}}(0)$ and $W^s_{\text{loc}}(0)$ up to cubic order.

   (b) Find the dynamics on the stable manifold up to cubic order.

   (c) Sketch the local phase portrait near $(0, 0)$.

3. (5 marks) Sketch a phase portrait in $\mathbb{R}^2$ which contains:

   - Three fixed points, one of which is a saddle, and two of which are asymptotically stable,
   - A homoclinic orbit
   - At least one periodic orbit.

   Label each of these objects clearly and draw a representative sample of other trajectories so the behaviour of solutions in all of phase space is clear.

4. (10 marks) The Lorenz equations are

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = rx - y - xz,$$
$$\frac{dz}{dt} = -bz + xy,$$

where σ, r and b are positive constants.

(a) For $0 < r < 1$, show that the origin is asymptotically stable.

(b) For the parameters $\sigma = 10$, $b = 8/3$, use XPP and any other methods to determine how the dynamics changes as the parameter r is varied in the interval $[0, 28]$. You should identify at least four parameter values at which there are qualitatively different types of flow. Include in your answer a phase portrait (plotted with XPP) for each qualitatively different type of behaviour you find. Describe the main changes that you observe.

5. * (10 marks) Consider the equations:

$$\dot{x} = -y - x^3,$$
$$\dot{y} = x^5.$$

(a) Find the Jacobian. What does this tell you about the stability of the equilibrium at the origin?

(b) Let $V = x^6 + 3y^2$. Compute $\frac{dV}{dt}$ along trajectories, and use this to show that the origin is asymptotically stable.

(c) What can you say about the basin of attraction of the origin?

(d) By considering the signs of $\dot{x}$ and $\dot{y}$, sketch the global phase portrait for this system. *You may use XPP to check your answer, but you must explain all your reasoning to get full marks.*