Maths 362 Lecture 1

Topics for today:
Partial derivatives and Taylor series (review of material from Maths 253)

Reading for this lecture: Greenberg Sections 13.3, 13.5

Suggested exercises: Greenberg Section 13.5: 1, 2, 9, 11

Reading for next lecture: Greenberg Sections 14.2-14.4

Today's handout: Course guide

Why study vector calculus?

The Navier-Stokes equations model fluid flow:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \nabla\cdot\mathbf{T} + f$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Here v is velocity of fluid (vector), p is pressure (scalar). First equation is Newton's law for fluid, second equation is conservation of mass.

Revision of some calculus for functions of several variables

Partial derivatives: Let f(x,y) be a function of variables x and y defined for (x,y) near a point (x_0, y_0).

Fix $y=y_0$. Then $f(x,y)=f(x,y_0)$ is a function of x alone.

The x-derivative of this function at x_0 (if it exists) is called the **partial derivative of f with respect to x at (x_0,y_0)**, and is written

$$\frac{\partial f}{\partial x} \quad \text{or} \quad f_x$$

There is a similar definition for the partial derivative w.r.t. y.

Example 1: Find partial derivatives w.r.t. x and y for $f(x,y)=x^3y^5$.

Example 2: Find the first partial derivatives of $f(x,y,z)=xy^2z^3$.

Partial derivatives may themselves be functions of the variables and we can take partial derivatives of these functions to get **second partial derivatives**:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \quad \text{etc.}$$

Example 3: Find the second partial derivatives of $f(x,y)=x^3y^5$.

The order of differentiation may matter. For example, maybe

$$f_{xy} \neq f_{yx}$$

for a particular function f.

However, if all the first and second partial derivatives exist and are continuous near (x_0,y_0) then

$$f_{xy} = f_{yx}$$

Taylor's formula and Taylor series

Let f(x) be a function of one variable x, with f′(x), f′′(x), … etc all existing. Then Taylor's formula is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^{n-1} + R_n(x).$$

$R_n(x)$ is the remainder term:

$$R_n(x) = \frac{f^{(n)}(\xi)}{n!}(x-a)^n$$

with ξ being a point in $[x,a]$.

Taylor's formula tells us we can approximate $f(x)$ by a polynomial of degree (n-1) with error bounded by $R_n(x)$.

Example 4: Find the Taylor formula up to terms of order two, for the expansion about $a = -1$ of

$$f(x) = \frac{1}{1+x^2}.$$

If the function f is infinitely differentiable then we can let $n \to \infty$ in the Taylor formula to get the **Taylor series**:

$$\mathrm{TS}f|_a = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j .$$

The Taylor series represents f if the series converges in some interval of x and if the function to which it converges is equal to f on some interval of x.

Taylor's formula and Taylor series can be defined for functions of more than one variable in a similar way.

For example, the Taylor series for $f(x,y)$ about (a,b) is:

$$
\begin{aligned}
f(x,y) = f(a,b) &+ f_x(x-a) + f_y(y-b) \\
&+ \frac{1}{2!}\left[f_{xx}(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2\right] + \dots.
\end{aligned}
$$

where all the derivatives are evaluated at (a,b).

Example 5: Find the Taylor expansion about (1,3,-2) for the function $f(x,y,z)=x^3yz$.

Important ideas from today:

- **partial derivatives**
- **second partial derivatives**
- **Taylor's formula for functions of one or more variables**
- **remainder terms**
- **Taylor series for functions of one or more variables**