Maths 190 Lecture 6

- **Topic for today: The irrational side of numbers**

- **Vitally important question of the day**:

  How many rational numbers are there?

What is a rational number?

- A rational number x is one that can be written in the form

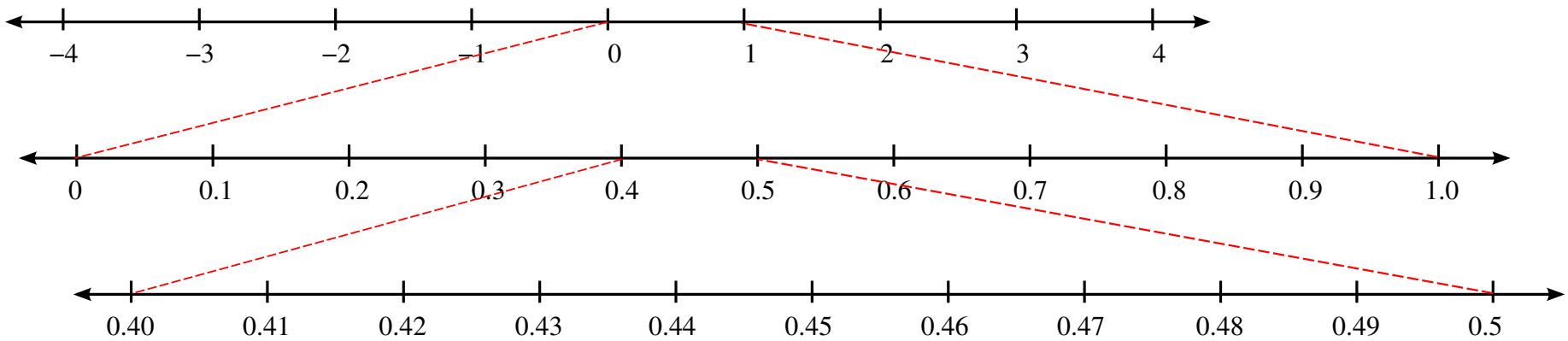
$$x = \frac{n}{m}$$

  for some integers n and m.
- If we can write a number as a terminating decimal, then it is rational

$$0.675 = \frac{675}{1000} = \ldots$$

- Rational numbers written as fractions can be reduced so that the top and bottom have no common factors.

How many rational numbers are there between 0 and 1?



- We can always find a rational number between two other rational numbers.
- If a and b are rational, then

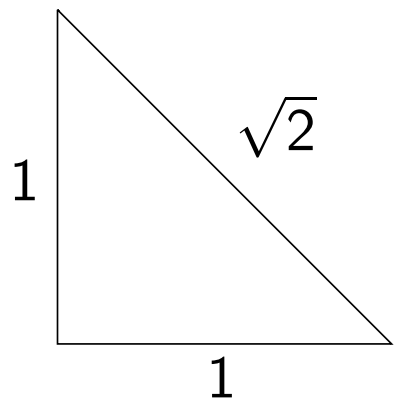
$$\frac{a+b}{2}$$

  is rational and between a and b.
- Check!

Are all numbers rational?

- The Ancient Greeks thought they were.
- Pythagoras' theorem (580 BC): sides a, b, c of a right angled triangle satisfy

$$a^2 + b^2 = c^2$$



- But what about $\sqrt{2}$?

Assume that $\sqrt{2}$ is rational.

- Then we can write
$$\sqrt{2} = \frac{a}{b}$$
where a and b are integers, with no common factors. In particular, they are not both multiples of 2 (even numbers).
- Let's square both sides to make it simpler:

- and multiply up....

- Since $a^2 = 2 \times b^2$, then a^2 must be an even number.
- So therefore a must be an even number.
- So let's write $a = 2 \times c$.
- Giving us...

- or, cancelling a 2 on each side

Hmmm....

- So now $b^2 = 2c^2$, so b^2 must be an even number.
- So therefore b must be an even number....

- *But we said a and b couldn't both be even numbers!*

- Contradition! $\sqrt{2}$ is *not* a rational number.

What about $\sqrt{3}$?

- Use the same ideas to prove that $\sqrt{3}$ is irrational.

Decimal expansions

- What does the decimal expansion of an irrational number look like?

$$\sqrt{2} = 1.414213562\ldots$$

- What does the decimal expansion of a rational number look like?

$$\frac{1}{2} = 0.5$$

$$\frac{1}{3} = 0.333333333\ldots$$

Periodic decimals

- Periodic decimals are rational:
  - Consider $W = 3.5959595959\ldots$
  - Then
  - So

Periodic decimals

- Rational numbers have periodic decimal expansions:

$$\frac{1046}{495} = 495\overline{)1046.0000\ldots}$$

A random number

- Pick a decimal number between 0 and 1 at random.
- What are the chances it is rational?

No holes, no neighbours

- Can you find an irrational number between two rational numbers? How can you find it?
- Can you find a rational number between any two irrational numbers? How?
- What's the rational number closest (but not equal to) zero?
- What's the irrational number closest to zero?

Important ideas from today:

- Some numbers are rational, others are not
- There are very few rational numbers compared to irrational numbers.
- But there are still infinitely many!

For next time

- Read Section 3.1 of the textbook. Beyond Numbers.