Maths 761 Lectures 18 and 19

Topic for Lectures 18 and 19
Homoclinic bifurcations in the plane

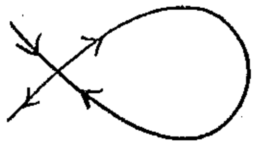
Recommended reading for Lectures 18 and 19:
Glendinning Chapter 12, to end of §12.3, excluding example using perturbation theory starting halfway down page 341)

Today's handouts
Lecture 18 and 19 notes

Homoclinic bifurcations in the plane

A solution $x(t)$ is *homoclinic to an equilibrium solution* x^* if $x(t) \to x^*$ as $t \to \infty$ and $x(t) \to x^*$ as $t \to -\infty$, and $x(t_0) \neq x^*$.

Homoclinic orbits of stationary solutions are rather special ('non-generic'). For instance, consider a flow in the plane, defined by $\dot{x} = f(x;\mu)$ where $x \in \mathrm{R}^2$, and assume there is a homoclinic orbit of a hyperbolic stationary solution x^* when $\mu = \mu_0$. Without loss of generality we can assume $x^* = 0$ and $\mu_0 = 0$. Then x^* must be a saddle with a one-dimensional stable manifold and a one-dimensional unstable manifold. One branch of the stable manifold and one branch of the unstable manifold coincide and form the homoclinic orbit, as shown in the diagram below.

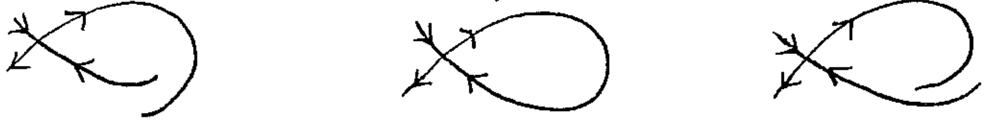


We would not in general expect two arbitrary one-dimensional manifolds to coincide in the plane; rather, we expect them to either miss one another completely or to intersect at isolated points (think of two lines in the plane). The stable and unstable manifolds in our example are solution curves so, by the Existence and Uniqueness Theorem, if they intersect at all they will be coincident. However, in general these two manifolds will not intersect. If we vary μ away from the value μ_0 we therefore expect the stable and unstable manifolds to separate, as shown in the diagram below.

$\mu<0$ $\mu=0$ $\mu>0$

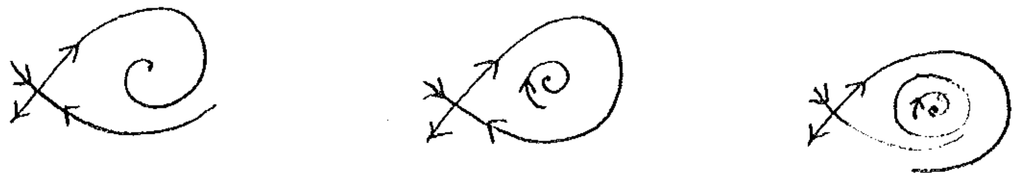
OR

$\mu<0$ $\mu=0$ $\mu>0$



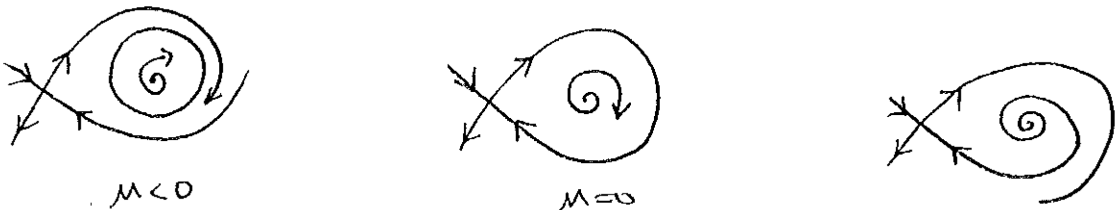
Without loss of generality, we assume that the first of the two possible sequences occurs.

The sequence of phase portraits shown cannot be complete; index theory tells us there is at least one stationary solution inside a homoclinic loop. Furthermore, if there is a single stationary solution inside the loop and it is hyperbolic when $\mu = \mu_0$, it must be a sink or a source, and will exist and be hyperbolic for μ in some interval about μ_0. A consequence of this is that in the previous sequence of pictures there must be one picture in which there is an annular region containing no stationary solutions and on the boundary of which all solutions cross inwards or all solutions cross outwards. By the Poincaré-Bendixson Theorem, there must be a periodic orbit in the annular region. Thus, for μ near μ_0 we expect to see the following sequence of phase portraits.



$\mu < 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu > 0$

OR



$\mu < 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu > 0$

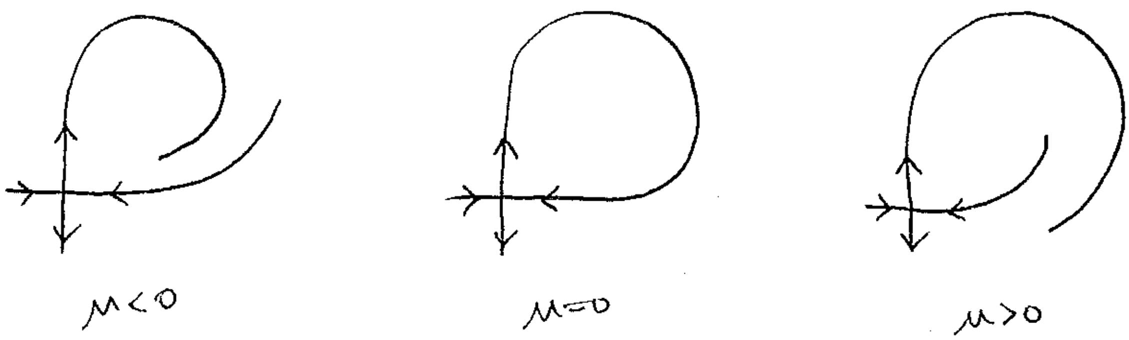
The qualitative change of behaviour seen in these sequences of phase portraits is called a *homoclinic bifurcation of a stationary solution.*

To summarise, a homoclinic bifurcation of a stationary solution occurs when the stable and unstable manifolds of the stationary solution pass through one another. A homoclinic bifurcation can be thought of as removing (or adding) one large amplitude, large period, periodic orbit from the dynamics when the periodic orbit collides with a saddle point. (The amplitude of the periodic orbit is 'large' in the sense that it doesn't tend to zero at the bifurcation, unlike the periodic orbit that appears in a Hopf bifurcation, and the period is 'large' because it tends to infinity as the bifurcation is approached, as will be seen later.)

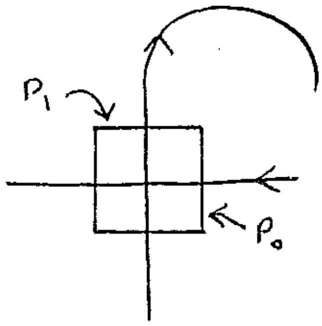
Constructing a geometric return map for a homoclinic bifurcation in the plane

We construct a map that models the main features of any generic planar flow that has a homoclinic bifurcation. The map will not depend explicitly on the details of any particular differential equation.

Consider a family of equations $\dot{u} = f(u; \mu)$, where $u \in \mathrm{R}^2$, $\mu \in \mathrm{R}$. Assume that there is a hyperbolic saddle point at the origin when $\mu = 0$, with eigenvalues $\lambda_s < 0 < \lambda_u$. Choose a coordinate system so that $W^u(0)$ and $W^s(0)$ are tangent to the coordinate axes at the origin. Assume that μ is chosen so that at $\mu = 0$ there is a homoclinic orbit of the saddle at the origin and that for μ near zero we see the sequence of (partial) phase portraits:



Choose $\epsilon > 0$ small and draw a box of side length 2ϵ about the saddle point. For μ near 0 we will have a picture something like the following.



We can now divide the flow into two parts, that inside the box and that outside the box.

Call the right side of the box P_0 and the top of the box P_1. For μ near 0 the flow defines two maps,

$$\tilde{\Sigma}_0 : P_0 \to P_1, \quad \tilde{\Sigma}_1 : P_1 \to P_0.$$

We construct maps Σ_0 and Σ_1 that closely approximate $\tilde{\Sigma}_0$ and $\tilde{\Sigma}_1$. For μ sufficiently close to zero, the composite map $\Sigma_1 \circ \Sigma_0$ will then be a good approximation to the true return map induced on the surface P_0 by the flow.

Inside the box: If ϵ is sufficiently small and if Poincaré's Linearisation Theorem applies, the flow inside the box can be written as the linear flow

$$\begin{aligned}\dot{x} &= \lambda_s x,\\ \dot{y} &= \lambda_u y,\end{aligned}$$

which has general solution

$$x(t) = x(0)e^{\lambda_s t},\ y(t) = y(0)e^{\lambda_u t}.$$

A solution curve that enters the box at $(x_0, y_0) = (\epsilon, y_0)$ for $y_0 > 0$ will therefore exit the box at (x_1, ϵ), where $x_1 = \epsilon e^{\lambda_s T}$ and T is the time spent inside the box. We find T by noting that the equation for y gives

$$\epsilon = y_0 e^{\lambda_u T}.$$

Solving for T gives

$$T = \frac{1}{\lambda_u}\ln\left(\frac{\epsilon}{y_0}\right),\quad y_0 \neq 0,$$

so

$$x_1 = \epsilon \exp\left(\frac{\lambda_s}{\lambda_u}\ln\left(\frac{\epsilon}{y_0}\right)\right) = \epsilon\left(\frac{\epsilon}{y_0}\right)^{\lambda_s/\lambda_u}.$$

Thus $\Sigma_0 : P_0 \to P_1$ is defined by

$$\Sigma_0(\epsilon, y_0) = \left(\epsilon\left(\frac{\epsilon}{y_0}\right)^{\lambda_s/\lambda_u}, \epsilon\right),\quad y_0 > 0.$$

Outside the box: The map $\Sigma_1 : P_1 \to P_2$ maps a point (x_1, ϵ) on P_1 to a point (ϵ, y^*) on P_2. If $\mu = 0$, there exists a homoclinic orbit and $\Sigma_1(0, \epsilon) = (\epsilon, 0)$, i.e. picking $x_1 = 0$ leads to $y^* = 0$. If μ and x_1 are non-zero but small compared with ϵ, we can treat the map Σ_1 as a perturbation of the map that occurs when $\mu = 0$; this allows us to expand Σ_1 in a power series in the two small quantities x_1 and μ. Hence,

$$y^* = c_0 + c_1 x_1 + c_2\mu + c_3 x_1^2 + c_4 x_1\mu + c_5\mu^2 + O(3).$$

Now, by inspection of the pictures at the beginning of this summary sheet, we see that:

1. $y^* = 0$ when $\mu = x_1 = 0$ so $c_0 = 0$;
2. $y^* > 0$ when $\mu < 0$ and $x_1 = 0$ so $c_2 < 0$;
3. $y^* > 0$ when $\mu = 0$ and $x_1 > 0$ so $c_1 > 0$.

Hence, to first order, $\Sigma_1(x_1, \epsilon) = (\epsilon, ax_1 + b\mu)$ where $a > 0$, $b < 0$.

The return map: An approximate return map is obtained by composing the maps Σ_0 and Σ_1:

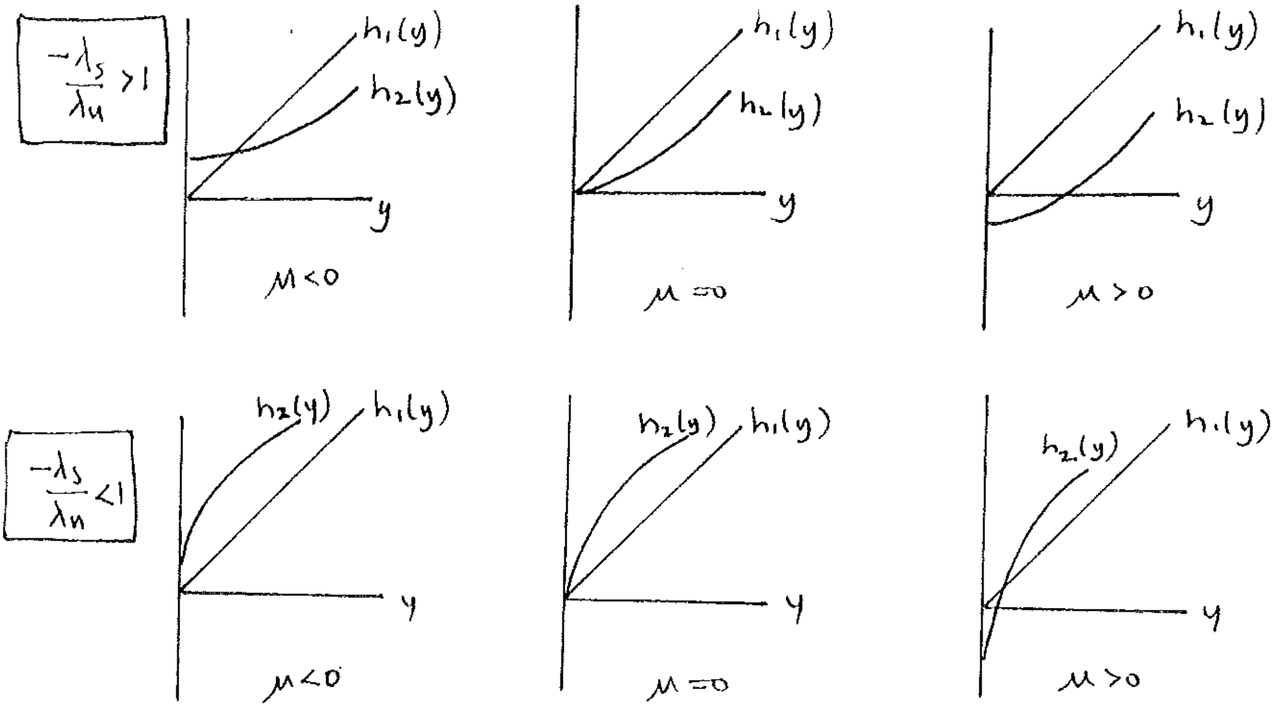
$$\begin{aligned}\Sigma_1 \circ \Sigma_0(\epsilon, y_0) &= \Sigma_1\left(\epsilon\left(\frac{\epsilon}{y_0}\right)^{\lambda_s/\lambda_u}, \epsilon\right) \\ &= \left(\epsilon,\ a\epsilon\left(\frac{\epsilon}{y_0}\right)^{\lambda_s/\lambda_u} + b\mu\right) \\ &= \left(\epsilon,\ k{y_0}^{-\lambda_s/\lambda_u} + b\mu\right)\end{aligned}$$

for some constants $k > 0$ and $b < 0$.

Periodic orbits in the flow: Periodic orbits in the planar flow will correspond to fixed points of the return map, i.e., to values of y for which

$$y = ky^{-\lambda_s/\lambda_u} + b\mu,\ k > 0,\ b < 0.$$

Sketching the functions $h_1(y) = y$ and $h_2(y) = ky^{-\lambda_s/\lambda_u} + b\mu$ (for $y > 0$) on the same axes gives:



We see that for $-\lambda_s/\lambda_u > 1$ the graphs of h_1 and h_2 intersect near $y = 0$ only if $\mu \leq 0$ with μ near zero. Furthermore, $|h_2'(y)| < 1$ at the point of intersection. We conclude that when $-\lambda_s/\lambda_u > 1$ there exists a stable periodic orbit in the flow for each value of μ near zero with $\mu < 0$. Similarly, if $-\lambda_s/\lambda_u < 1$, there is an unstable periodic orbit in the flow for each value of μ near zero with $\mu > 0$.

The period of the periodic orbit: The period (τ)of the orbit found above is equal to the time spent inside the box (τ_0) plus the time spent outside the box (τ_1). The time spent inside the box is

$$\frac{1}{\lambda_u}\ln\left(\frac{\epsilon}{y_0}\right).$$

Now $y_0 \to 0$ as $\mu \to 0$ so we can expand y_0 as a series in μ, i.e. $y_0 = d_1\mu + d_2\mu^2 + \ldots$ To first order, the time spent inside the box is then

$$\tau_0 = \frac{1}{\lambda_u}\ln\left(\frac{\epsilon}{|d_1\mu|}\right)$$

or

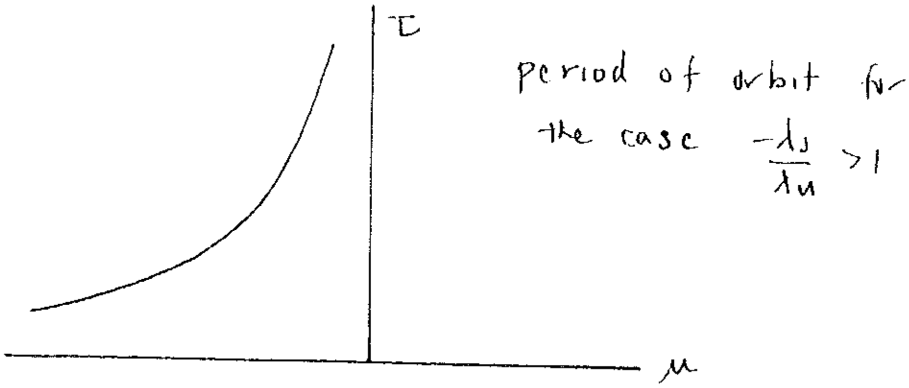
$$\tau_0 = \text{constant } -\frac{1}{\lambda_u}\ln|\mu|.$$

The time the periodic orbit spends outside the box is approximately the time $\tilde{T}$ spent outside the box by the homoclinic orbit, i.e.

$$\tau_1 = \tilde{T} + f_1\mu + \ldots$$

The total period of the periodic orbit is then

$$\tau = \tilde{T} + \text{constant } -\frac{1}{\lambda_u}\ln|\mu| + O(\mu).$$



In the limit $\mu \to 0$, the period of the orbit tends to ∞ as $-1/\lambda_u \ln|\mu|$. Thus, the growth of the period depends on λ_u not λ_s, and does not depend on ϵ (as is sensible). We see again that a homoclinic orbit can be thought of as a periodic orbit with infinite period.

Grand summary

Assume that a planar system of ODEs has a homoclinic orbit to a hyperbolic saddle when $\mu = \mu^*$ and that the eigenvalues of the saddle are λ_s and λ_u with $\lambda_s < 0 < \lambda_u$. Then:

1. For $\mu \approx \mu^*$ (but for only one sign of $\mu - \mu^*$) there is a periodic orbit in the flow.

2. If $-\lambda_s/\lambda_u > 1$ the periodic orbit is stable and if $-\lambda_s/\lambda_u < 1$ the periodic orbit is unstable.

3. As $\mu \to \mu^*$, the periodic orbit gets close to the saddle at one point on the orbit, and the period, τ, of the orbit tends to infinity, with scaling $\tau \sim \text{constant} - \frac{1}{\lambda_u}\ln|\mu - \mu^*|$.

Example: Bogdanov-Takens equations

The system

$$\begin{aligned} \dot{x} &= \lambda + x + y^2 + xy \\ \dot{y} &= x \end{aligned}$$

has a saddle at $(0, \sqrt{-\lambda})$, and the saddle has a homoclinic bifurcation at $\lambda \approx -1.76$. The eigenvalues of the Jacobian evaluated at the saddle at the bifurcation are 3.17 and -0.84. Since $-\lambda_s/\lambda_u < 1$ we expect an unstable periodic orbit to be created/destroyed in the homoclinic bifurcation. This is what we see with XPPAUT.