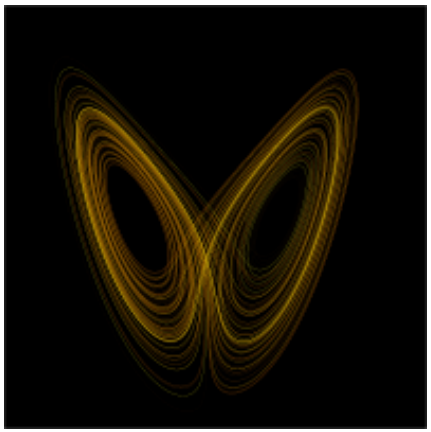
Maths 190 Lecture 18

Topic for today:



Chaos!!!

Periodic and aperiodic fluctuations

Question for today:

Why can we predict the tides but not the weather?

Many systems show repetitive behaviour

Some examples:

- Tides
- Motion of planets
- Rush hour

Does round-off error matter?

- What is $\sqrt{2}$ on your calculator?
- When might the round off error matter?

- A simple rule:
  1. Pick a number between 0 and 1.
  2. Multiply by 180
  3. Press 'SIN'. Write down the answer.
  4. Repeat 2 and 3.
- Pick a number half way down your list and try again. Do you get the same numbers?

Duelling calculators

- With your neighbour, pick the same number and repeat the previous exercise.
- Do your calculators agree?

Small changes can make big changes!

- Butterfly effect
- Effects predicitions about planetary motion, weather patterns and other complex systems.

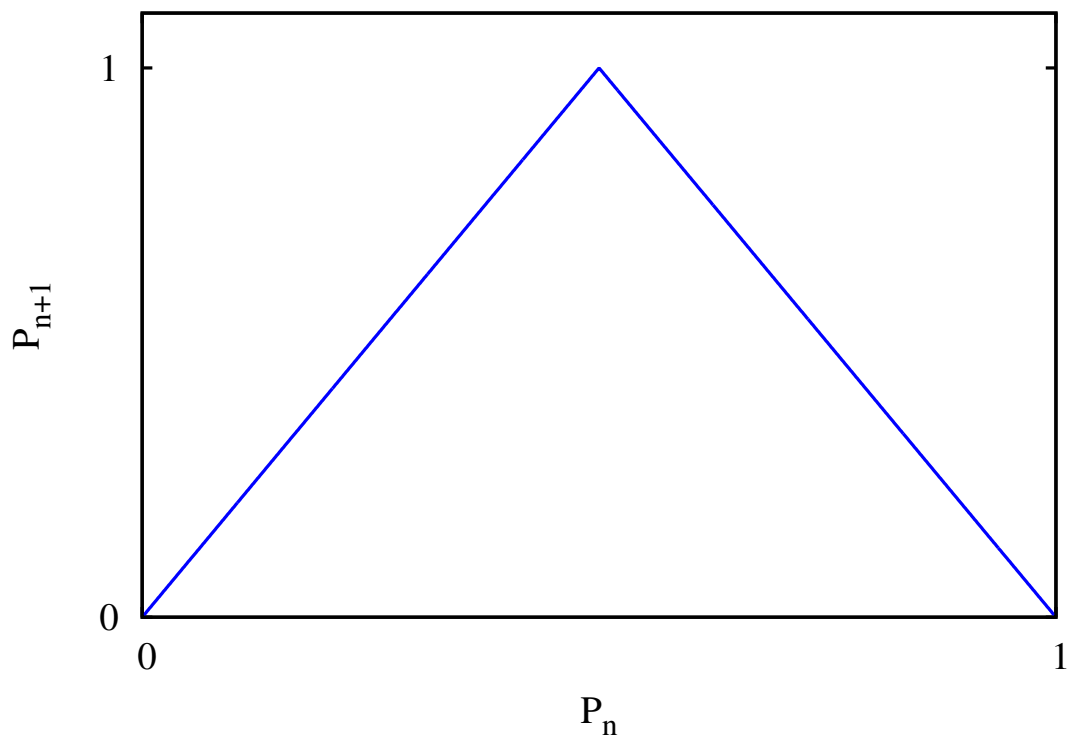
Folding pastry

- A piece of pastry is folded in half and back over itself, then stretched to the original length:

$$f(x) = \begin{cases} 2x \text{ if } x \in [0, \frac{1}{2}] \\ 2 - 2x \text{ if } x \in (\frac{1}{2}, 1] \end{cases}$$

The “pastry folding” map

- Take **a point** P_0... which **moves to** $f(P_0)$ after folding...
- while $f(P_0)$ is moved to $f(f(P_0))$
- **Simplification:** Travel via diagonal



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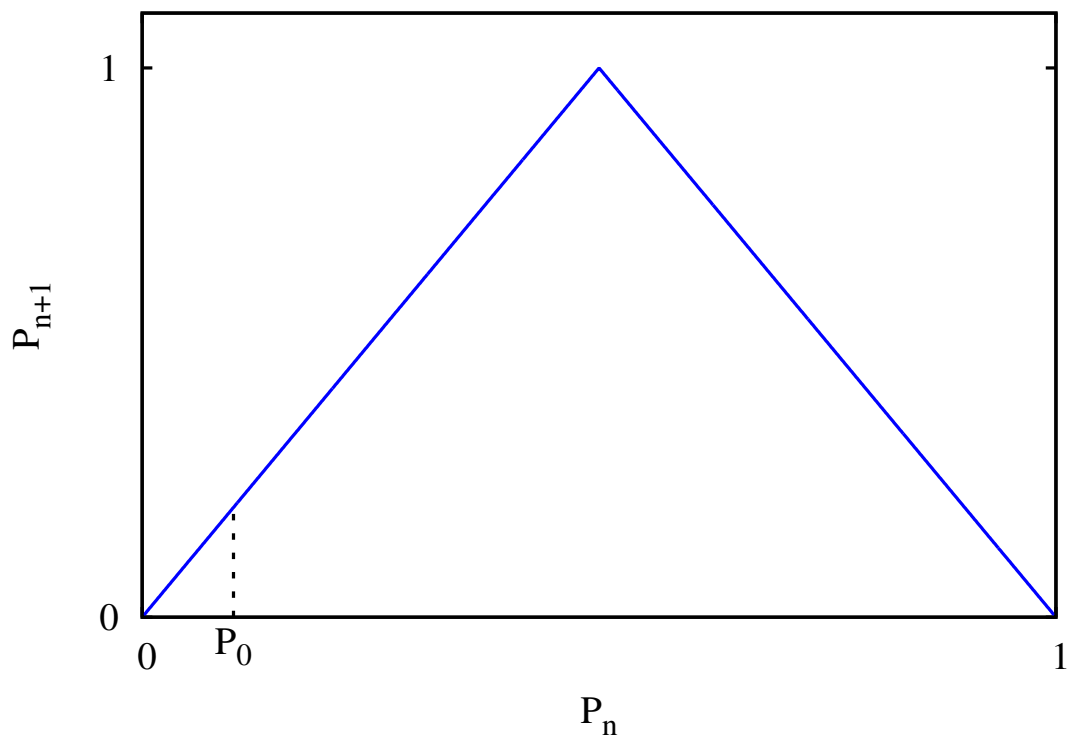

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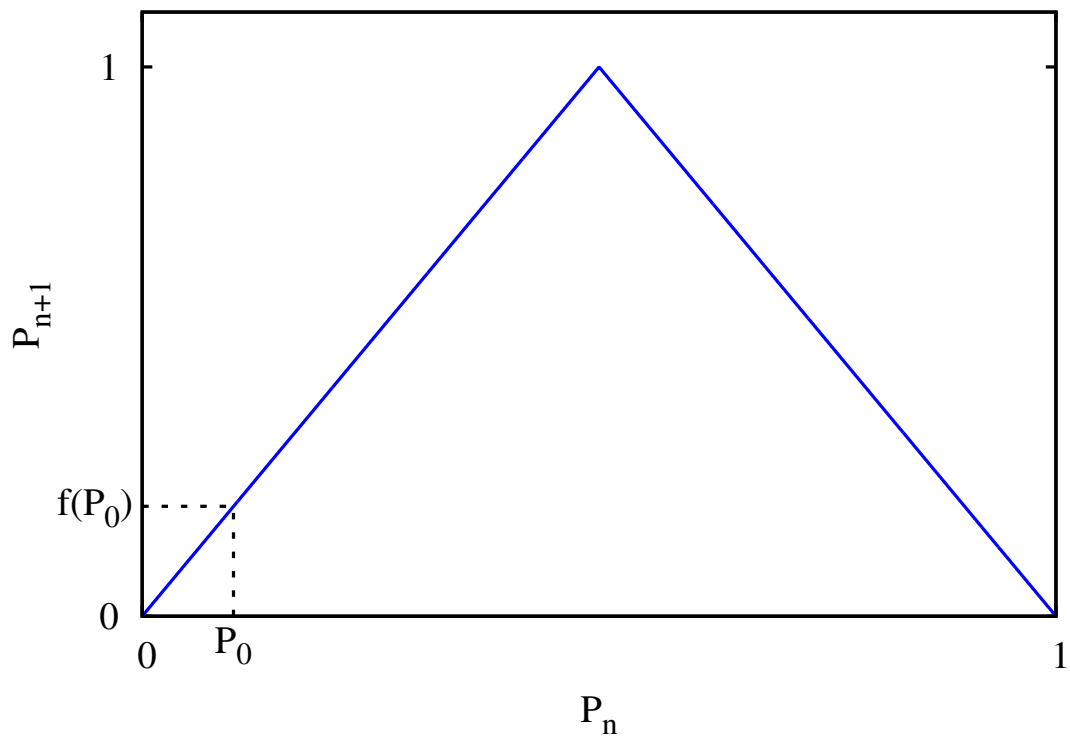

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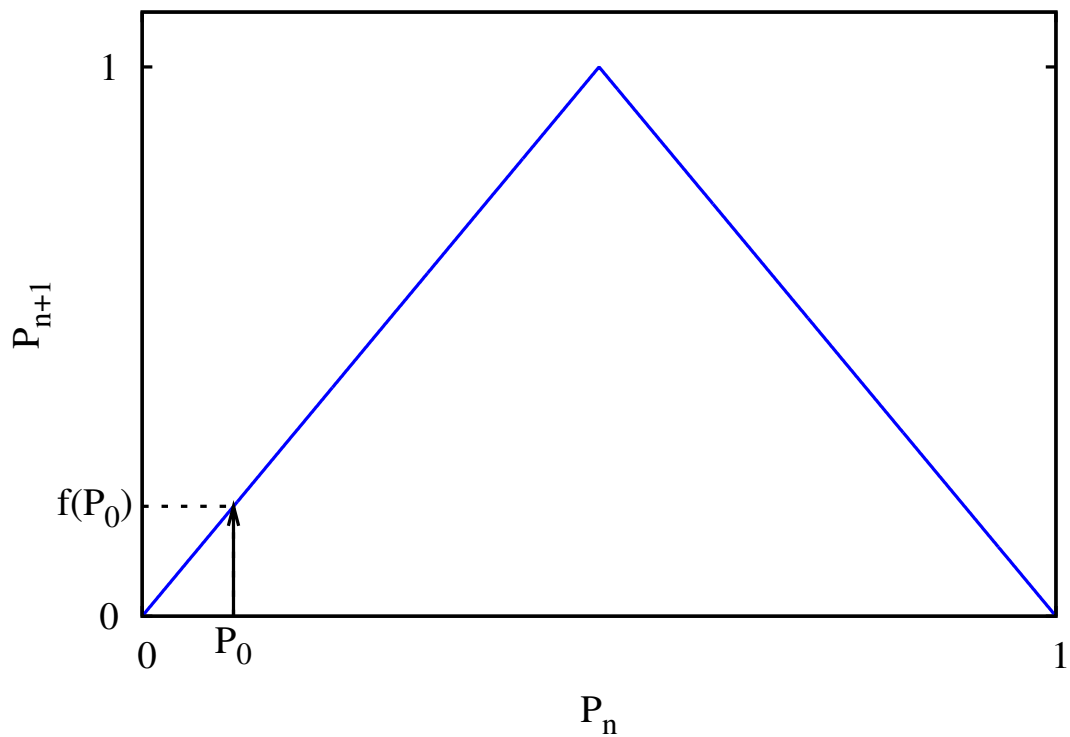

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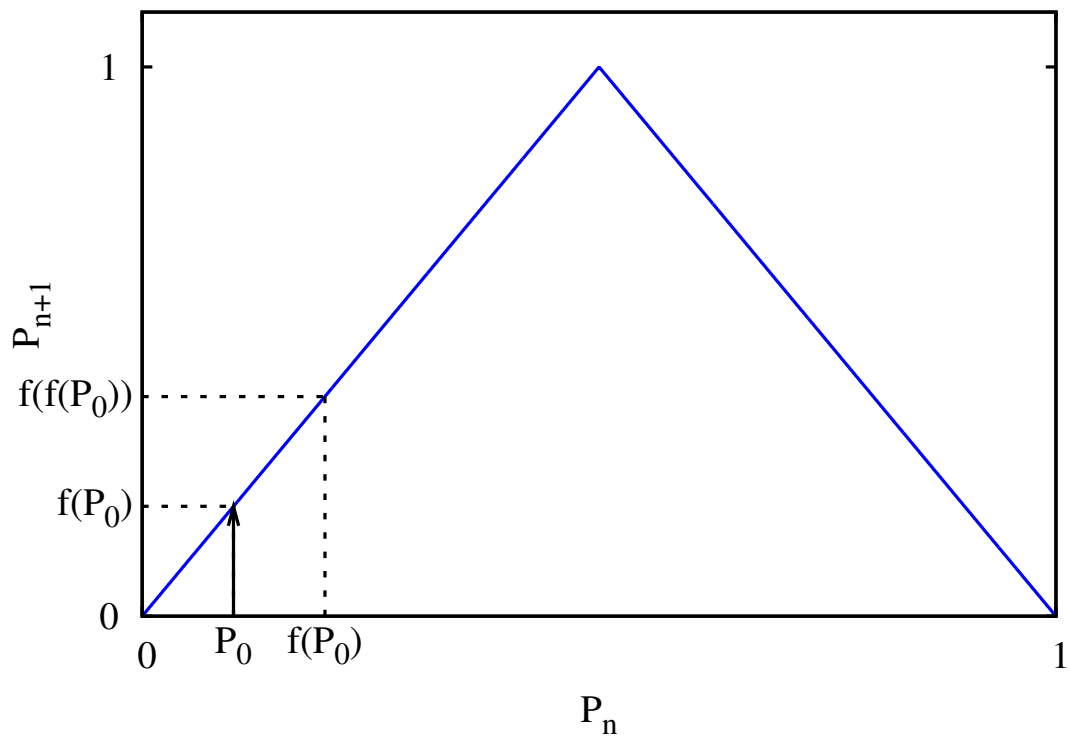

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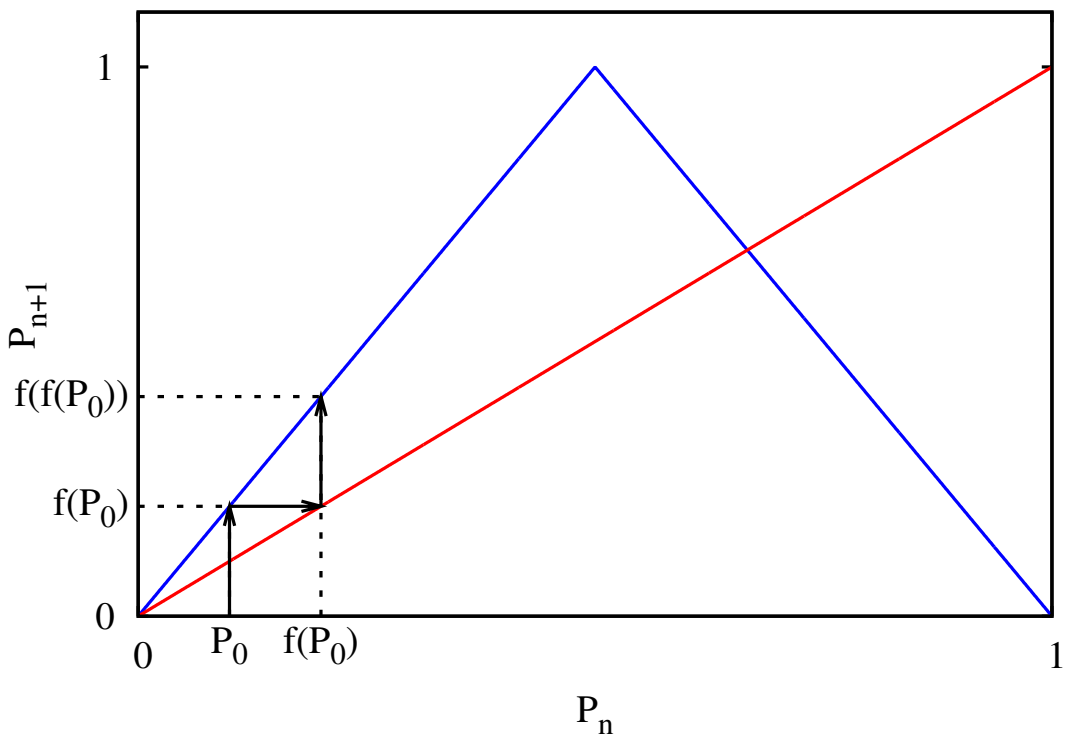

The "pastry folding" map

Question:

Have you seen someting like the "pastry folding" map before?

Answer:

The "pastry folding" map is an **iterative dynamical system:**

$$P_{n+1} = f(P_n) = \begin{cases} 2P_n \text{ if } P_n \in [0, \frac{1}{2}] \\ 2 - 2P_n \text{ if } P_n \in (\frac{1}{2}, 1] \end{cases}$$

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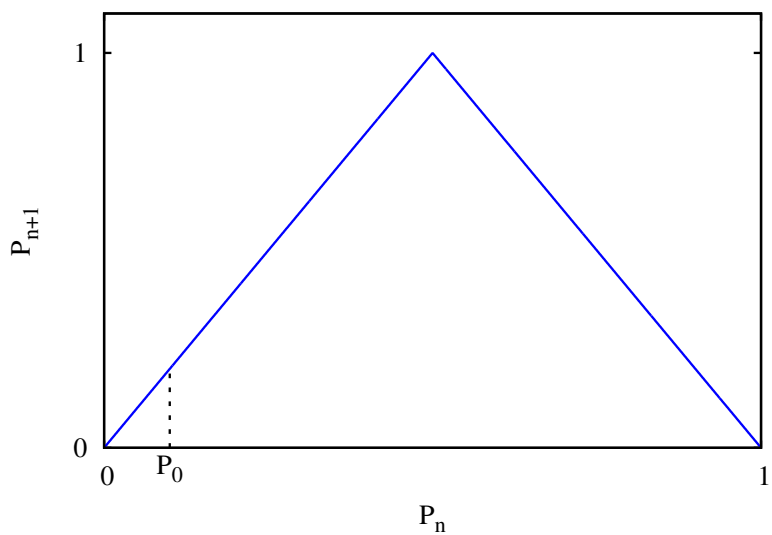
Following points:

Start at the point $(P_0, 0)$.

- Travel vertically to the blue line.
- Travel horizontally to the red line.
- Repeat.



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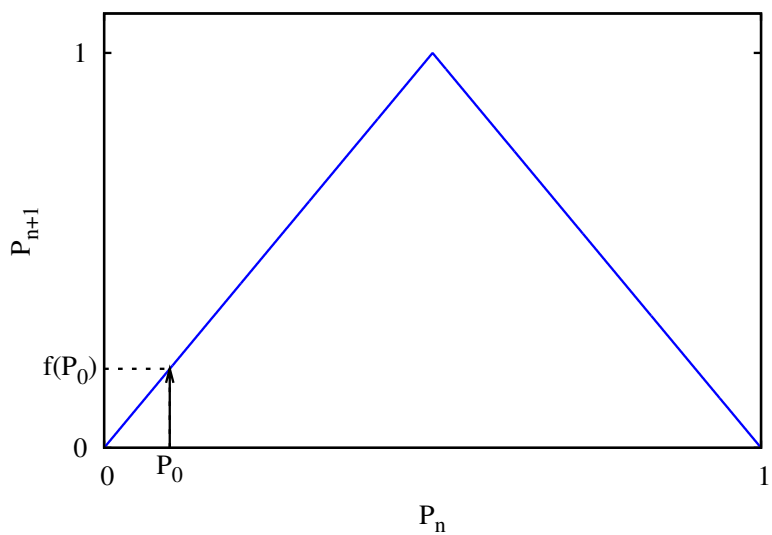

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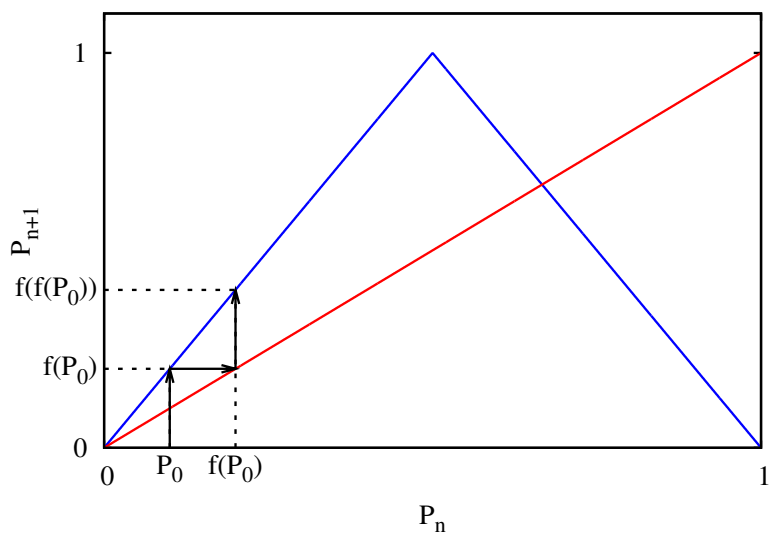

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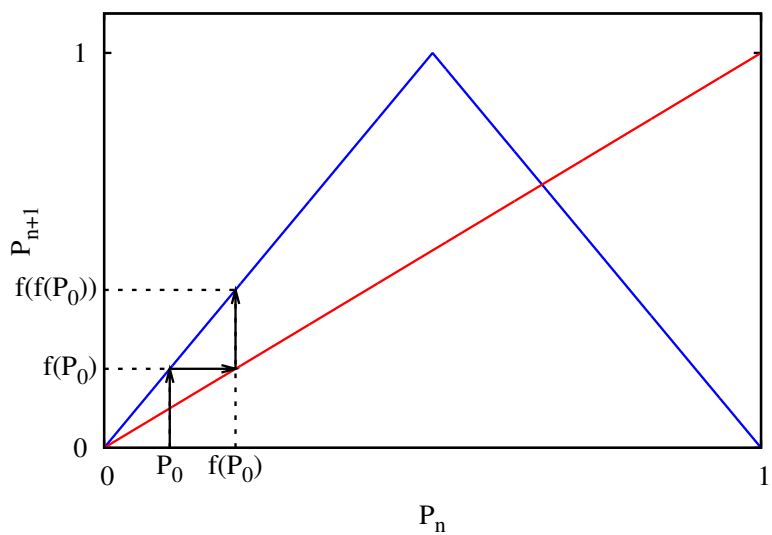
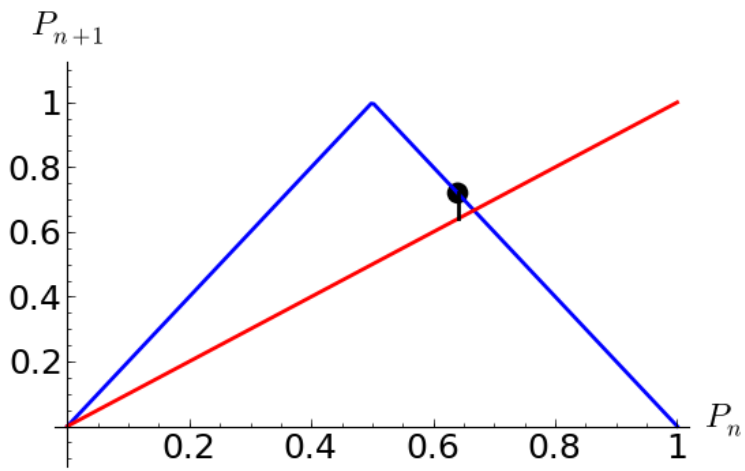


Cobwebs

- We can understand **iterative dynamical systems** *without calculations!*
- These plots are also called **cobweb plots**...
- ... any ideas, why?



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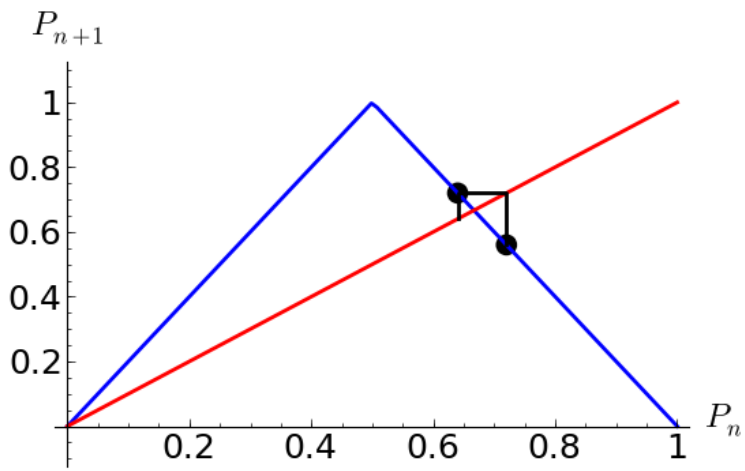

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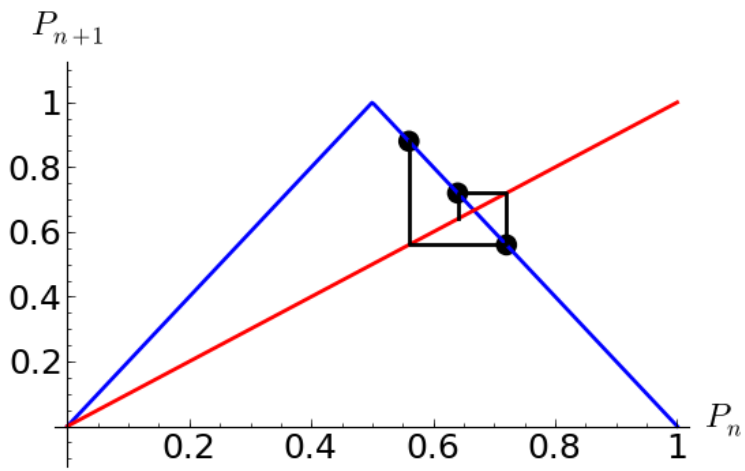

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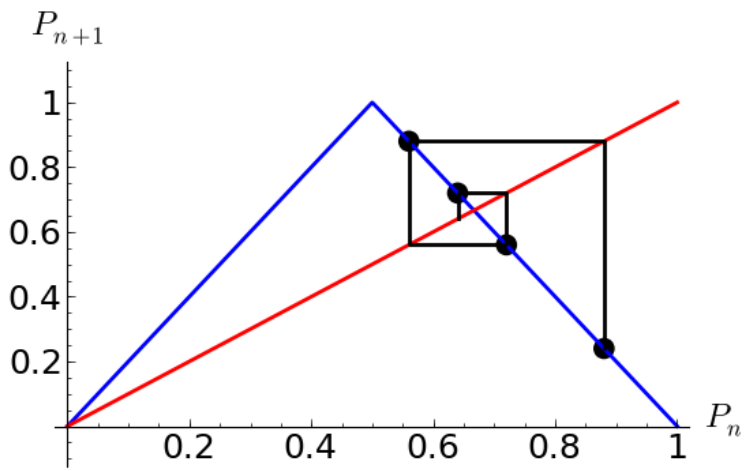

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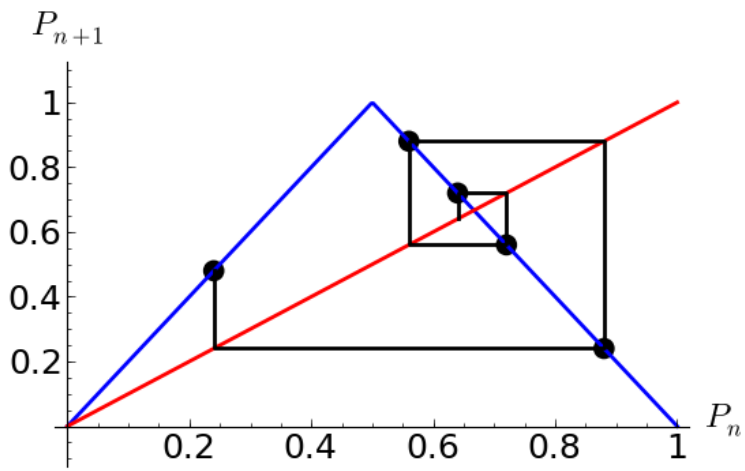

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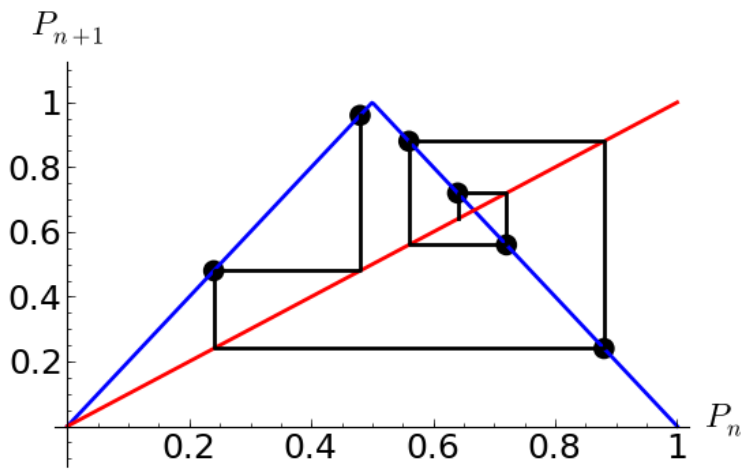

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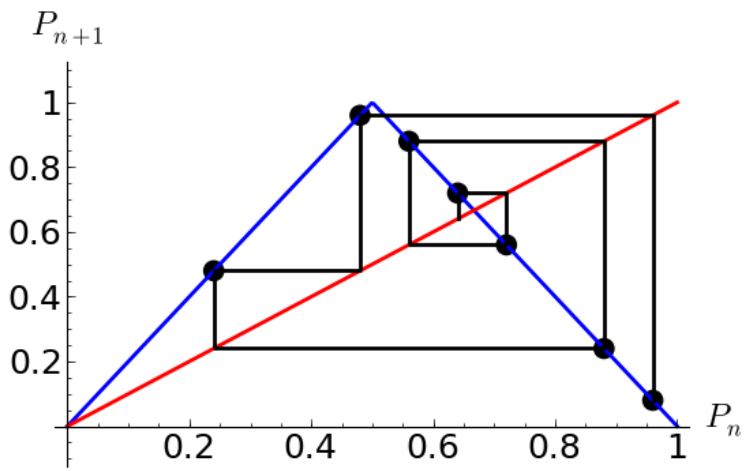

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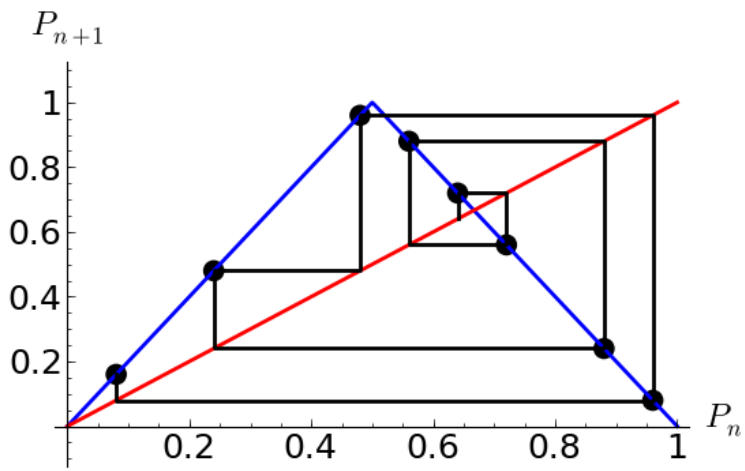
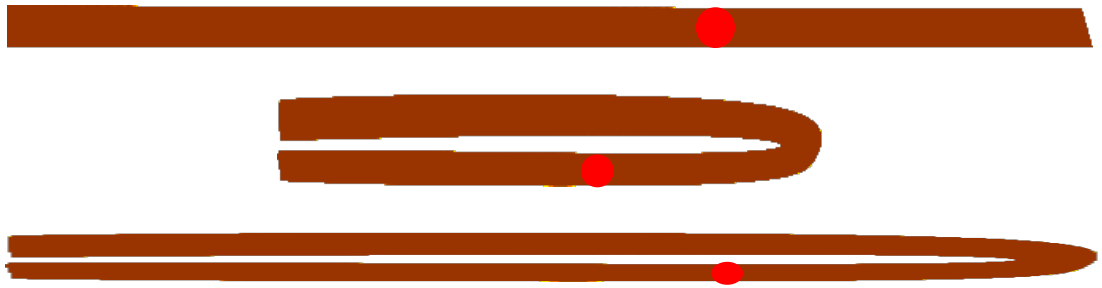
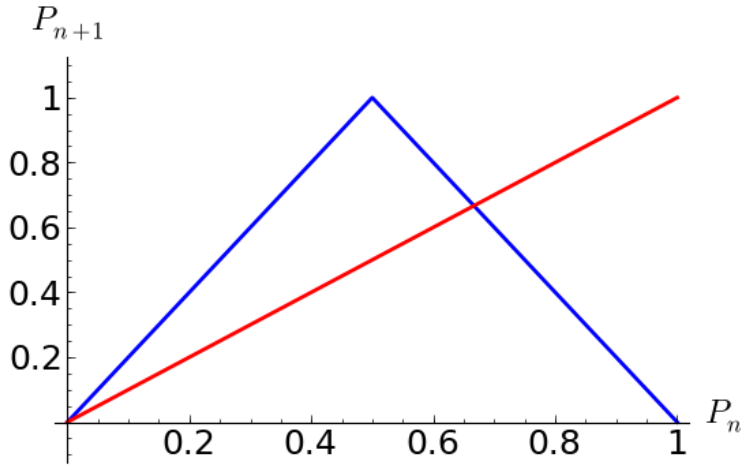


Fixed points



- Can you find a **point which does not move**?
- Yes... at the **intersections** of blue and red line



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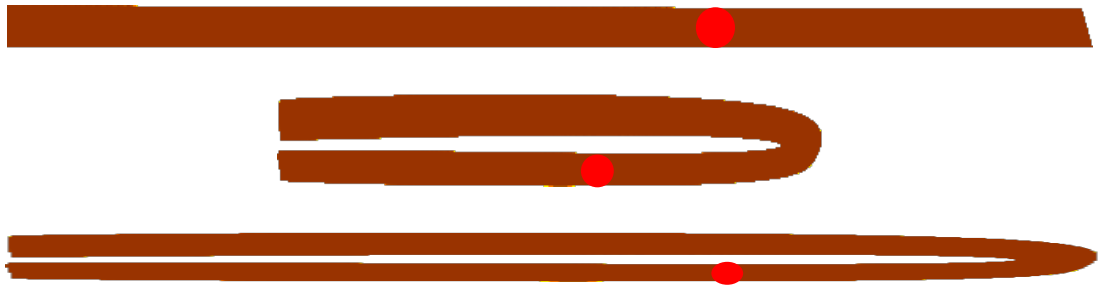

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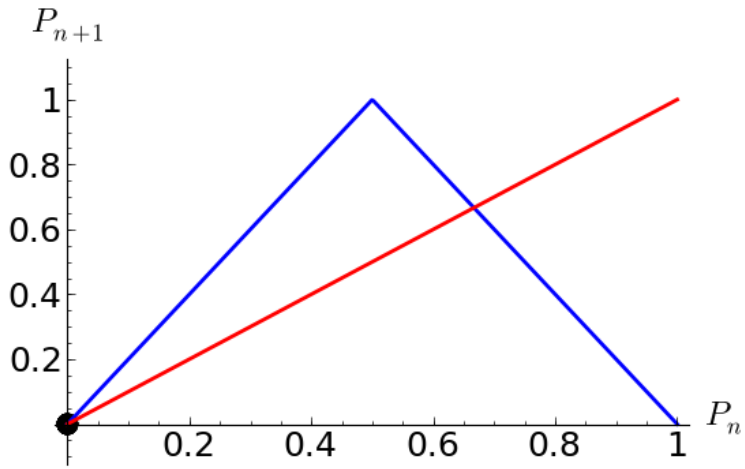

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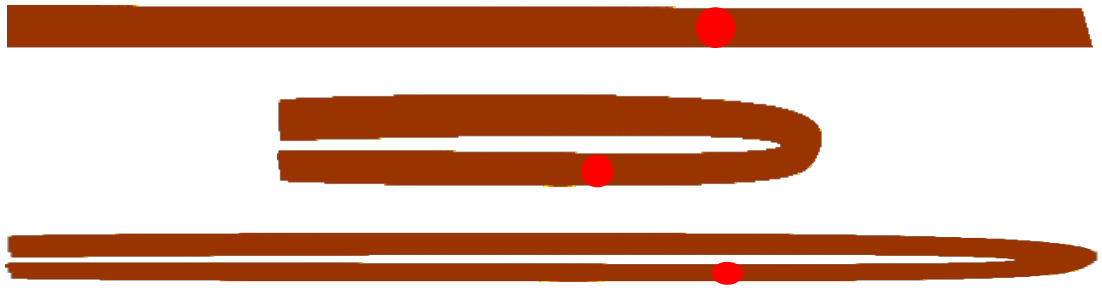

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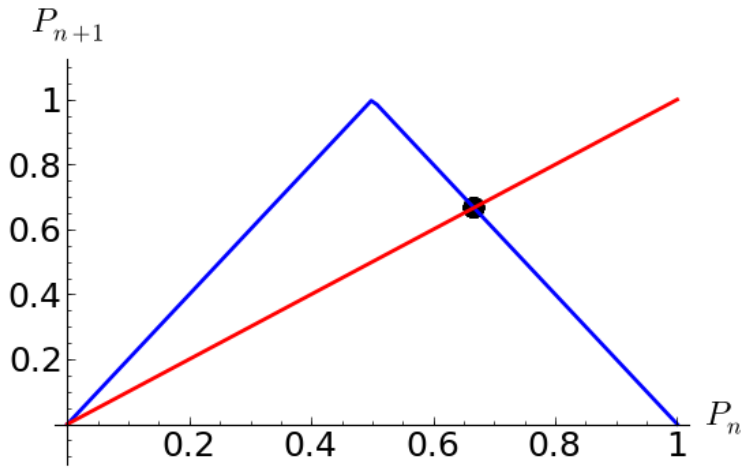

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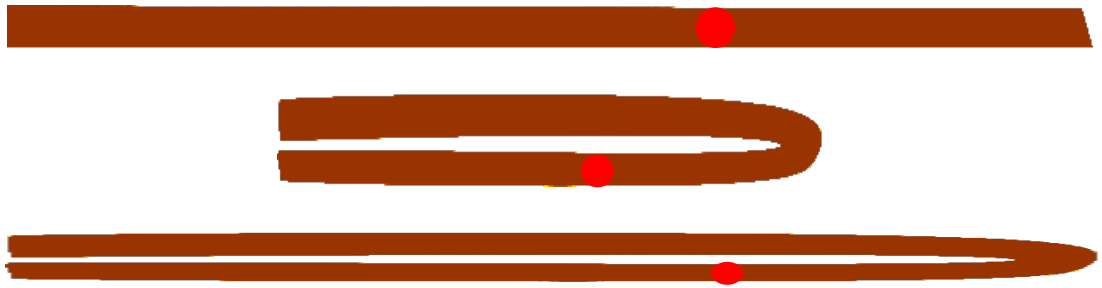

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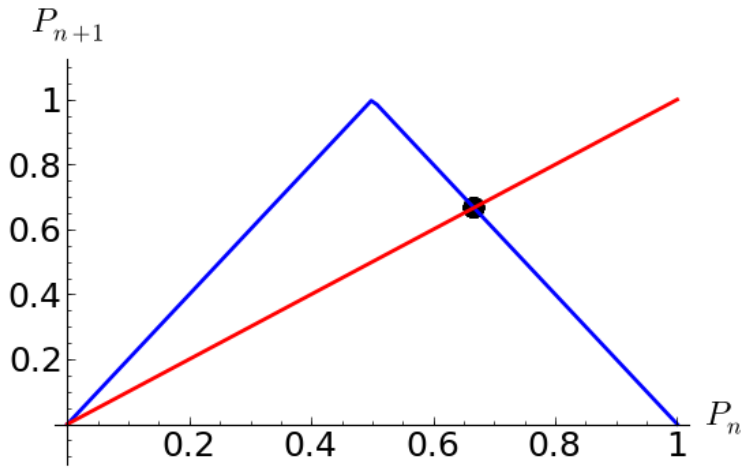
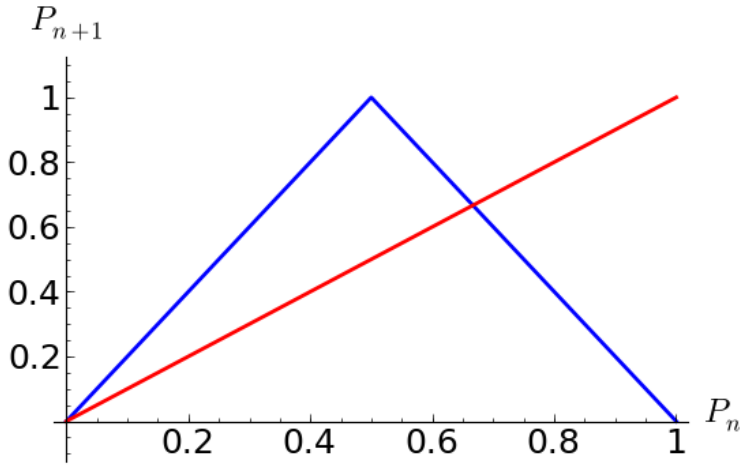


Simple periodic points

- A **rectangle** in the cobweb plot—points jump between two positions...
- These are the **simple periodic points**



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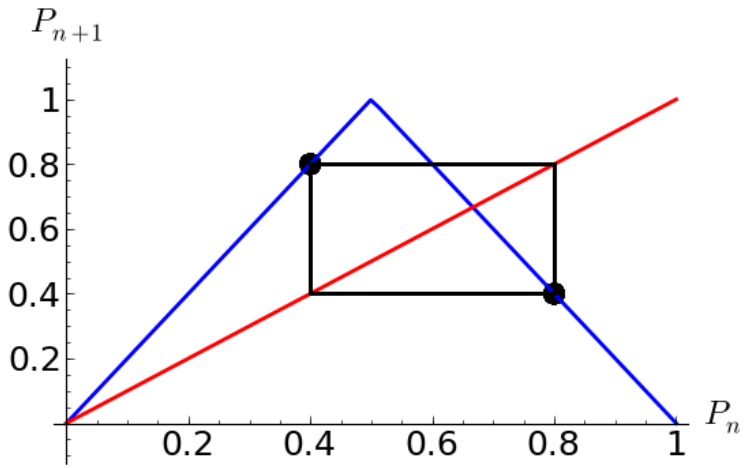

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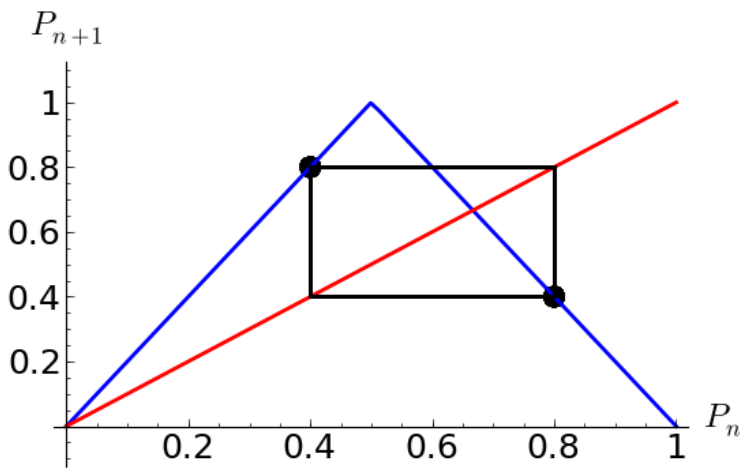
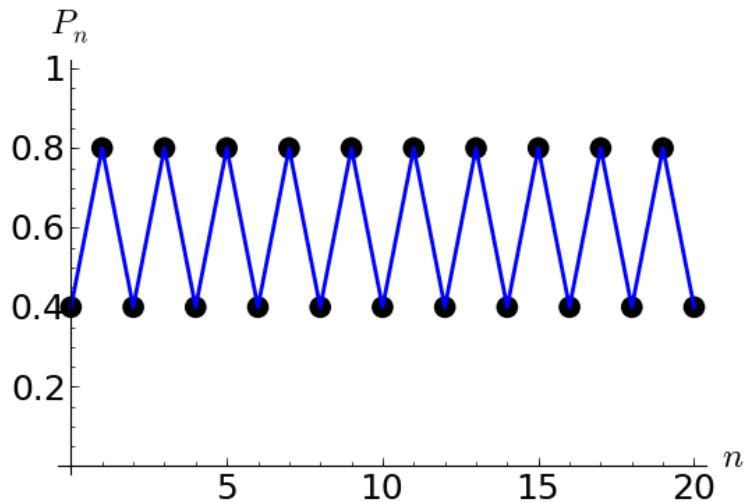
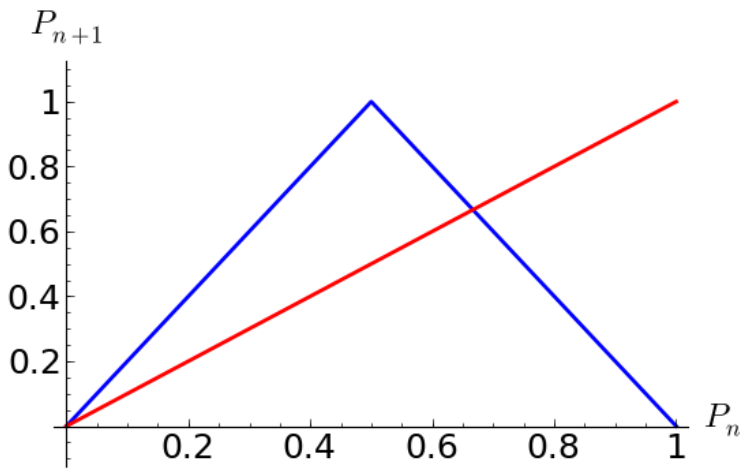




Period 3 points

- Can we **jump between three points?**
- Yes, we can!



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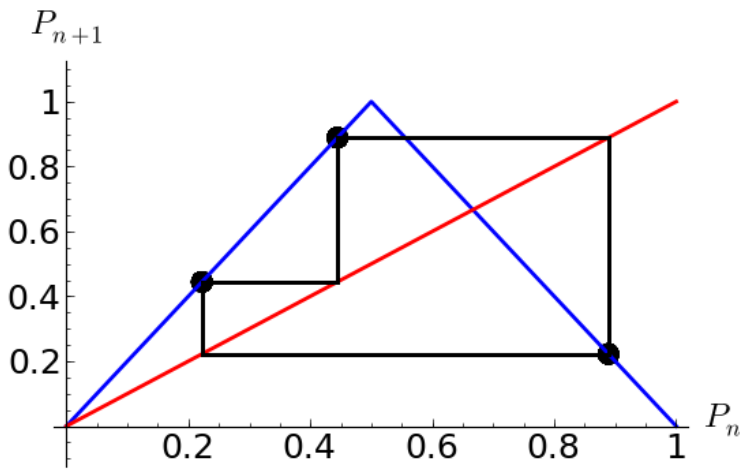

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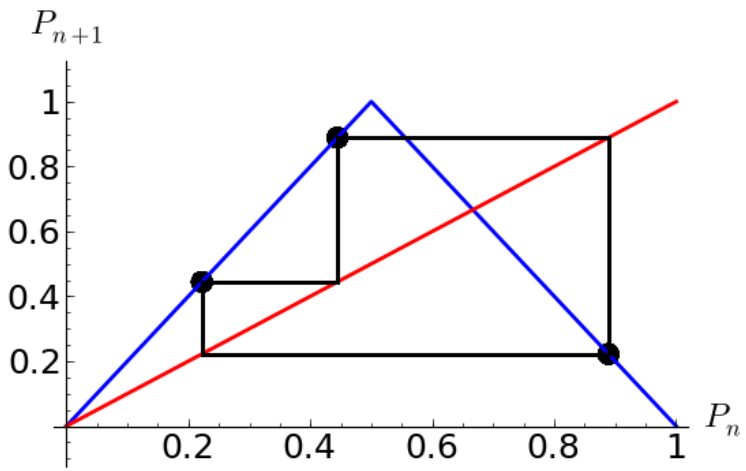
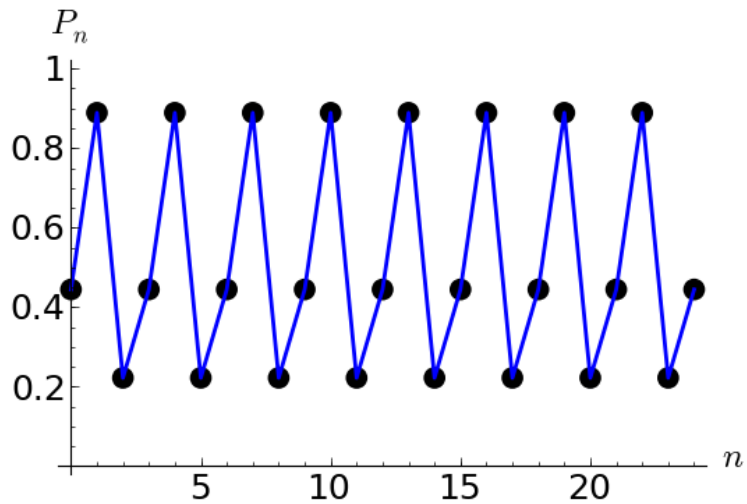
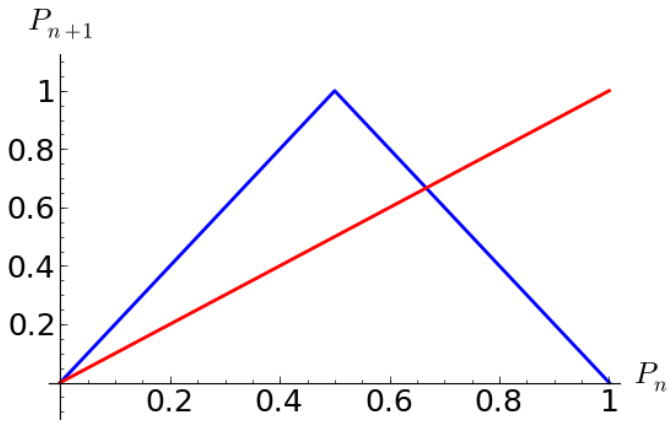




Period n points?

- Can we find cycles of arbitrary long periods?
- The answer is...



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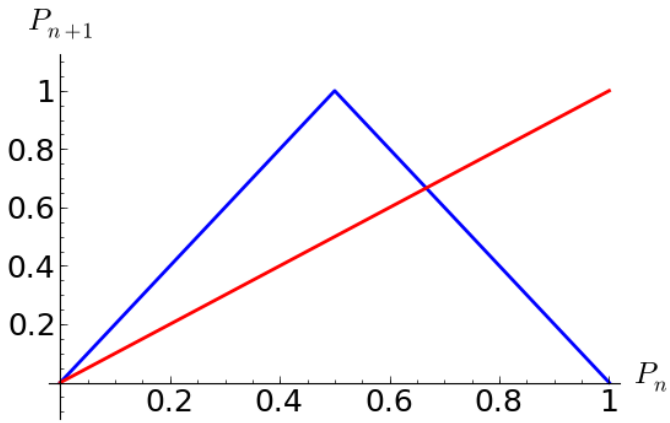

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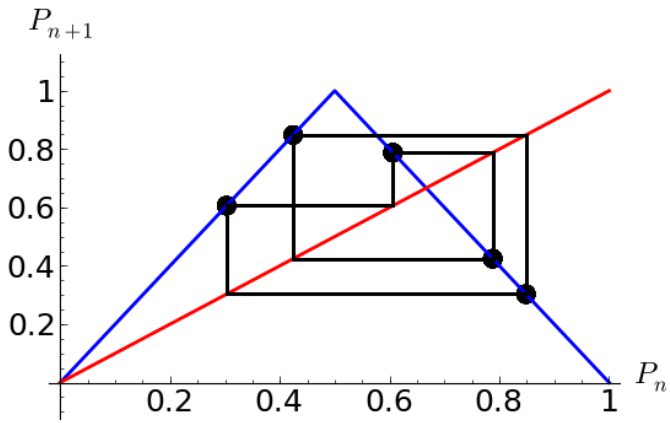

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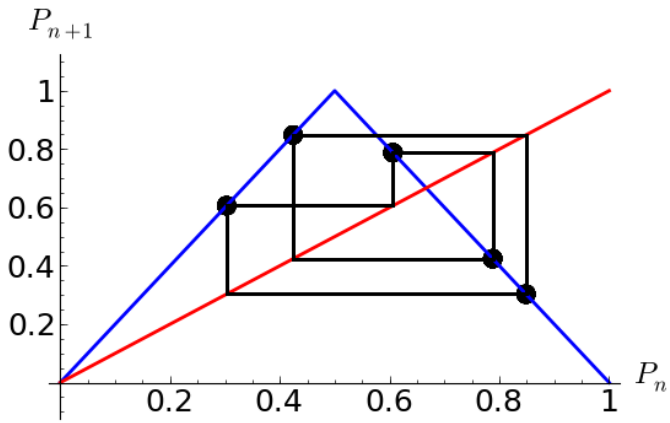
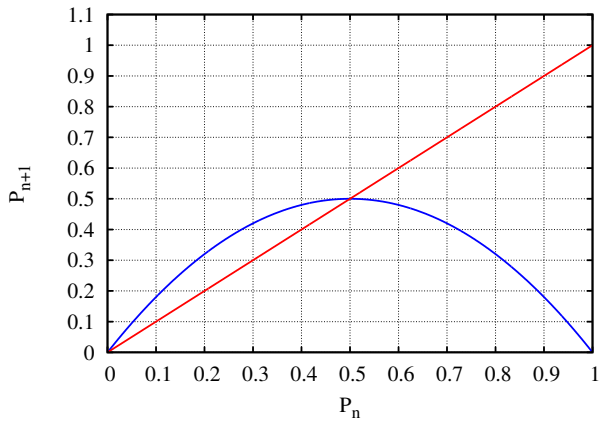


How can these points be calculated from the map?

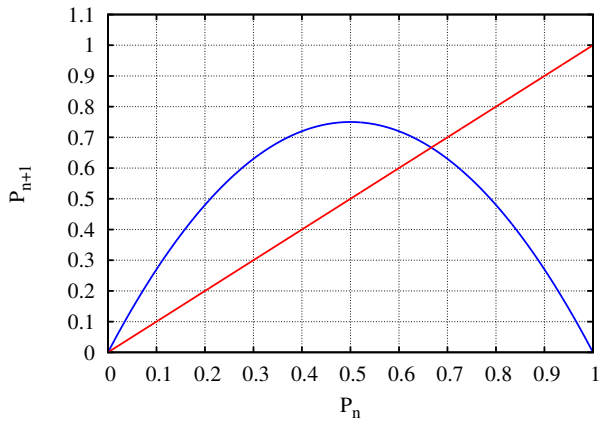
$$P_{n+1} = \begin{cases} 2P_n \text{ if } P_n \in [0, \frac{1}{2}] \\ 2 - 2P_n \text{ if } P_n \in (\frac{1}{2}, 1] \end{cases}$$

Now, it's your turn!

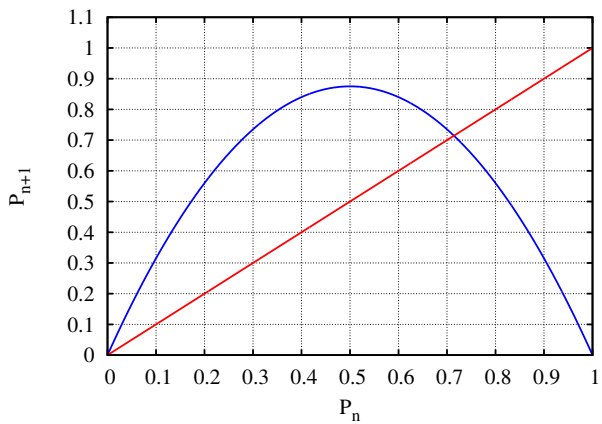
Try the **slightly less simple population model:**



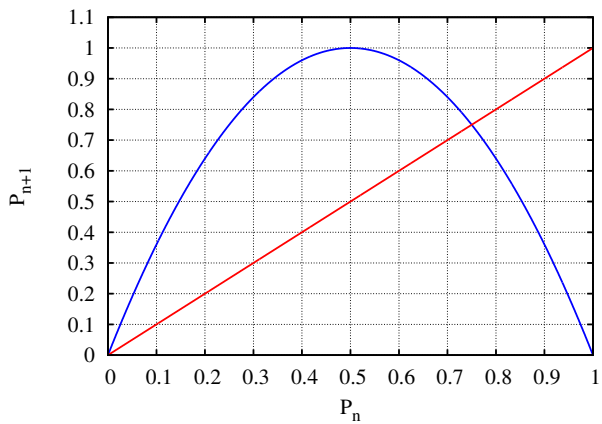
(a) $r = 2$



(b) $r = 3$



(c) $r = 3.5$

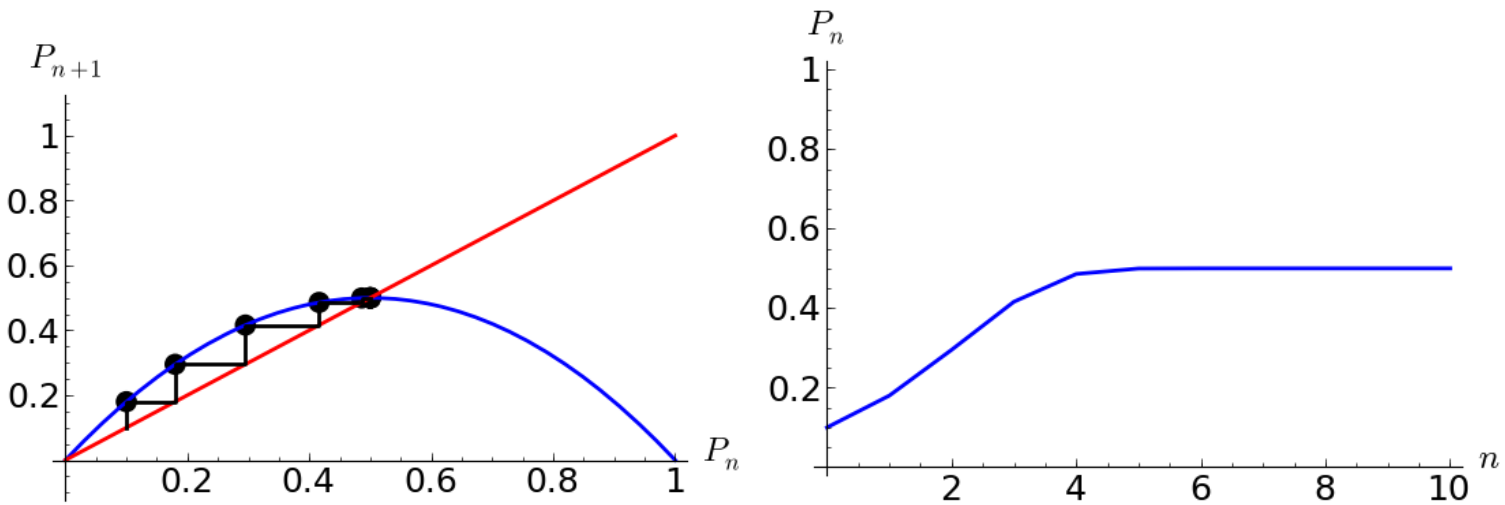


(d) $r = 4$

The logistic map

This population model is also known as the **logistic map**

$r = 2$: **Limited growth, fixed point:**



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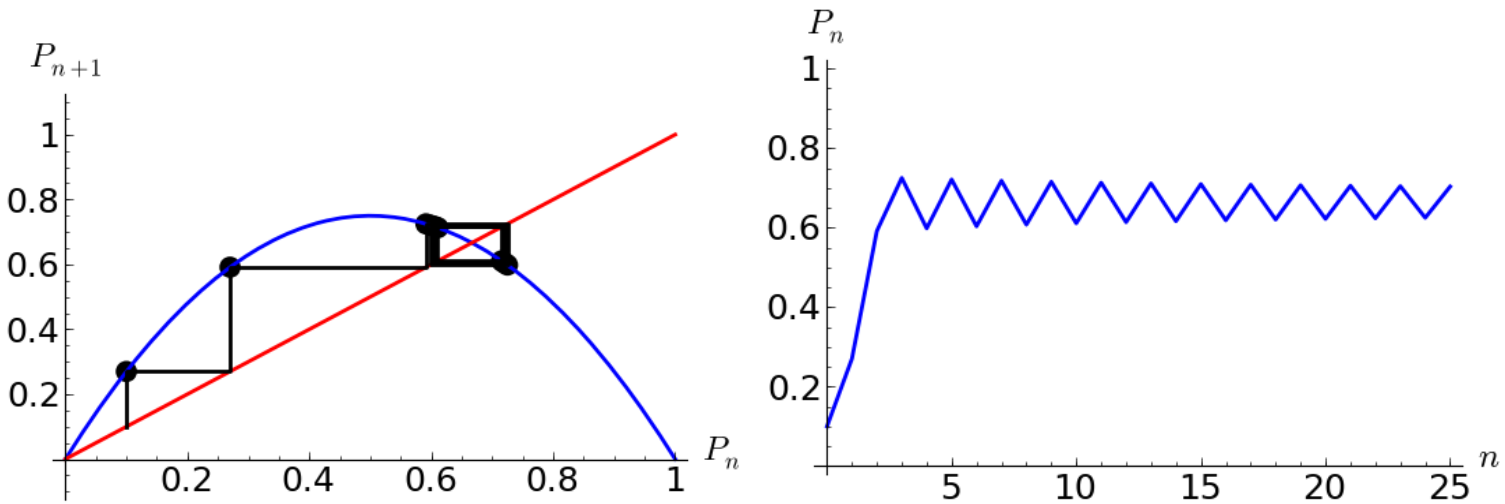
$r = 3$: **Periodic:**



The logistic map

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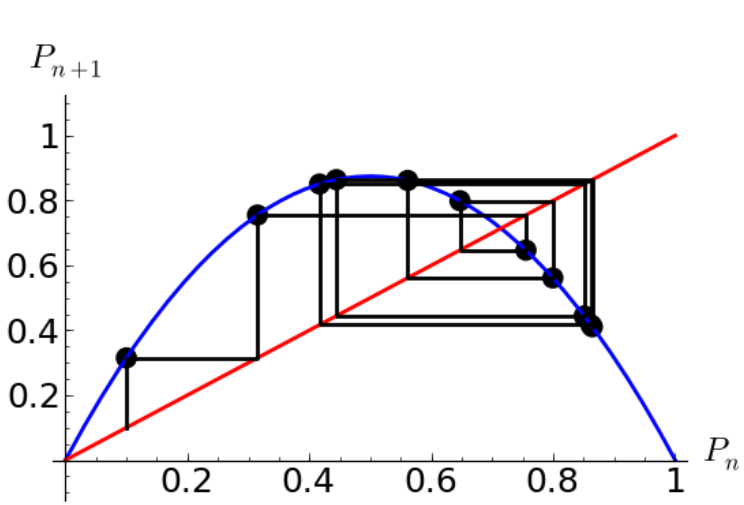
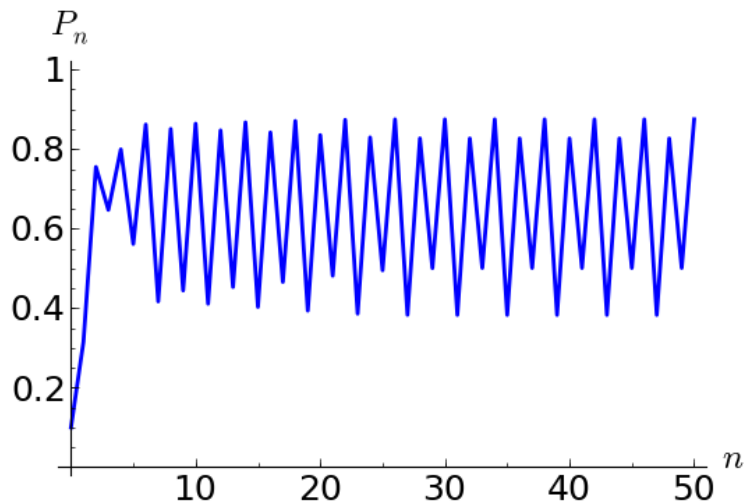
$r = 3.5$: **4-periodic:**





The logistic map

This population model is also known as the **logistic map**

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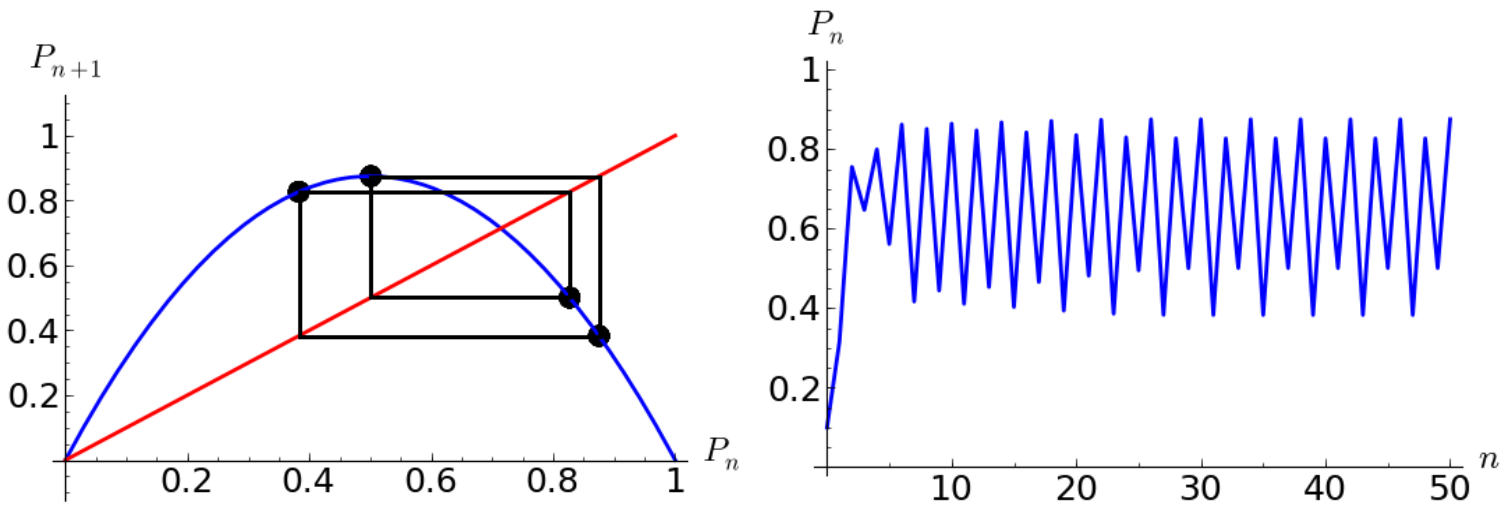

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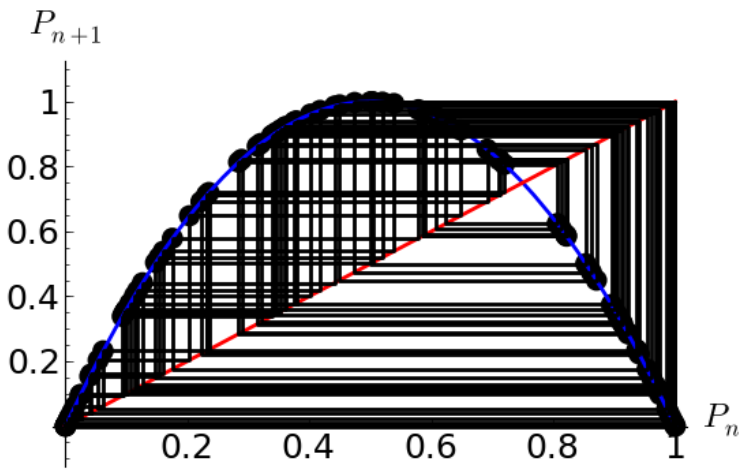
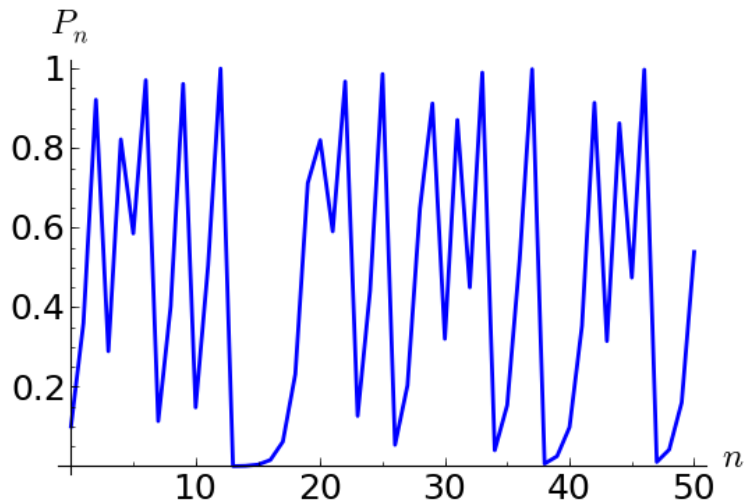
$r = 4$: **Chaos:**





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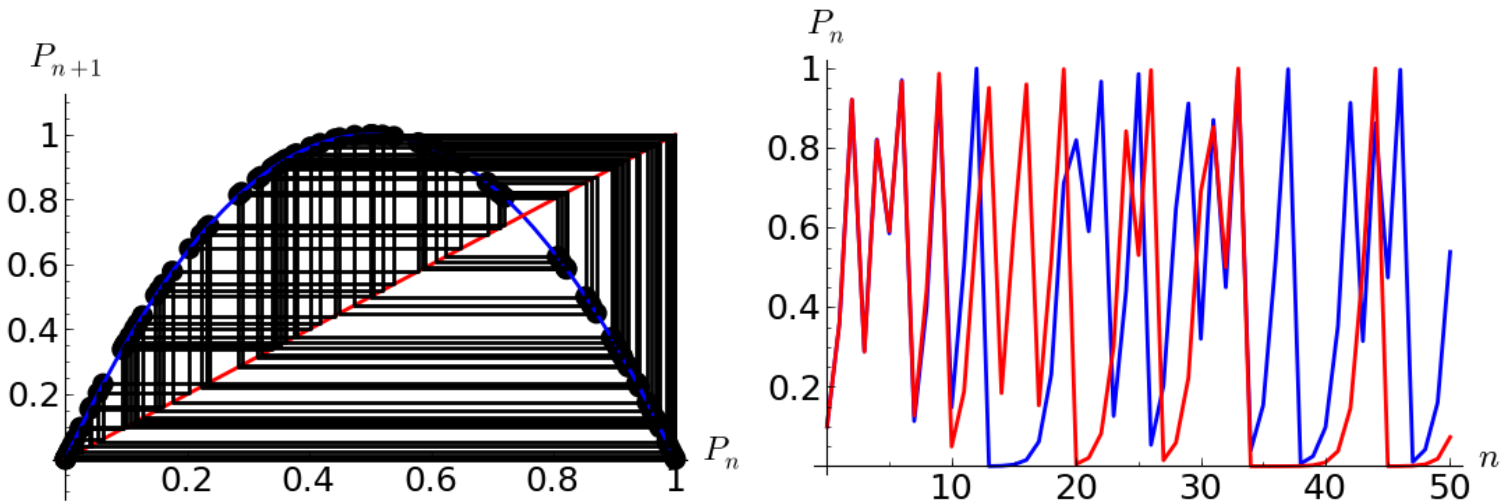

Chaos—mathematically defined!

Surprisingly, it is possible to find a good mathematical definition for **chaotic dynamics**.

Definition (Chaos)

The behaviour of a system (e.g. an *iterative dynamical system* or a *cellular automaton*) is **chaotic** if it meets all of the following criteria:

1. It does **not grow beyond all boundaries** (like, for example, the *Bank account*/*the simple population model*; some initial conditions of the *Game of Life*.
2. It is **not periodic**, we also say **aperiodic**.
3. It **depends highly sensitively on the initial conditions**.

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Important ideas from today:

- Repeating **simple rules** can lead to **unexpected complex behaviour.**
- A **small change** in where you start can lead to **huge differences** in where you end up.

For next time

- Read section 5.1