THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2009
Campus: City

MATHEMATICS

Great Ideas Shaping our World

(Time allowed: TWO hours)

NOTE: This paper contains **5** questions. Answer **ALL** questions. Show **ALL** your working. There are 100 marks available in total.

1. (a) (5 marks)

(i) Vivien has a pile of chocolate fish in her office which she agrees to share between herself, Claire and Rod. What is the smallest number of fish she needs to ensure that, however the fish are distributed between the three of them, at least one of them gets at least 5 fish?

(ii) It turns out that Claire doesn't actually like chocolate fish, and so they only have to be divided between Vivien and Rod. Assuming the number of fish in Vivien's office is the exact number you gave for part (i), what is the largest number n that always satisfies the statement *at least one of Vivien and Rod has at least n fish*?

(b) (11 marks)

(i) Write down the first 12 terms of the Fibonacci sequence and describe how it is constructed, either in words or using a formula.

(ii) What is a prime number? Give two examples of numbers that are prime and two examples of numbers that are not prime.

(iii) If I gave you a 6-digit number, describe in words how you would determine whether or not it was prime.

(iv) *Fibonacci primes* are numbers which are both Fibonacci numbers and prime numbers. Find the first five Fibonacci primes.

(c) (4 marks)

(i) Is $\sqrt{4}$ rational or irrational?

(ii) Vivien tries to use the following argument to prove that $\sqrt{4}$ is irrational. Explain where her mistake is.

A. Assume $\sqrt{4}$ is rational, then I can write it as a fraction $\frac{a}{b}$ where a and b do not have any common factors.

B. I simplify this to find $a^2 = 4b^2$, which means a^2 is a multiple of 4, so a must be a multiple of 4.

C. So I can write $a = 4c$.

D. Which gives me $16c^2 = 4b^2$ or simplifying, $4c^2 = b^2$.

E. But this means b^2 is a multiple of 4, so b must be a multiple of 4.

F. But we said a and b didn't have any common factors, but they are both multiples of 4.

G. Contradiction! Hence $\sqrt{4}$ is irrational.

2. (a) (2 marks) Explain what is meant by the statement "Set A has the same cardinality as set B".

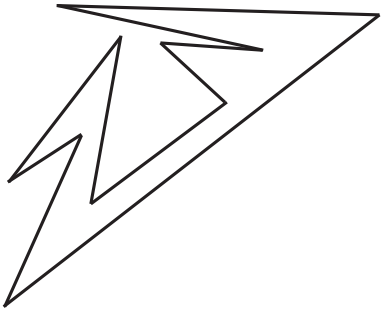
(b) (4 marks) Carefully show that the set of all odd positive integers $O = \{1, 3, 5, 7, \dots\}$ has the same cardinality as the set of natural numbers $N = \{1, 2, 3, 4, 5, \dots\}$.

(c) (9 marks) Let D be the set consisting of all numbers between 1 and 2 which have decimal expansions that use only the digits 1 and 2. So, for example, the number 1.1212121212.... could be in the set D but the number 1.123123123..... is not. Carefully show that the cardinality of D is greater than the cardinality of the set of natural numbers.

(d) (5 marks) Carefully show that the set of points on a circle of radius 1, excluding a single point, has the same cardinality as the set of points on the real line.

3. (a) (8 marks)

(i) The picture below shows the floor plan for an Art Gallery. Find the minimum number of guards required so that when the guards are placed at appropriate vertices, each point of the gallery can be viewed by at least one guard. On the special coloured answer sheet provided, show one way in which this number of guards could be placed so that each point of the gallery can be seen by at least one guard.



(ii) Is it possible to draw the floor plan of an Art Gallery with 10 straight sides so at least four guards, standing at vertices, are required to ensure that each point of the gallery can be viewed by at least one guard? Give a reason for your answer.

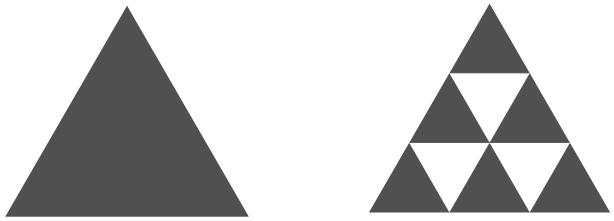
(b) (12 marks)

(i) Carefully explain the difference between **rigid symmetry** and **symmetry of scale** for patterns in the plane. Illustrate your explanation with appropriate pictures.

(ii) Sketch an example of a pattern in the plane that has rigid symmetries but no symmetry of scale.

(iii) Is it possible to have a pattern in the plane with symmetry of scale but no rigid symmetries? Give a reason for your answer and illustrate your answer with appropriate pictures.

4. (a) (10 marks) The zeroth and first stages in the construction of a fractal are shown below.



(i) What is the linear scaling factor between the large triangle in the zeroth stage, and one of the smaller triangles in the first stage?

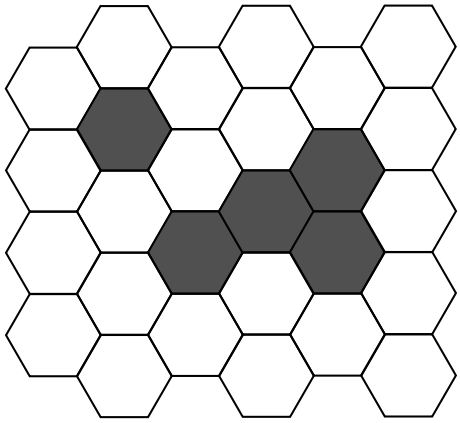
(ii) If the area of the original triangle is 1, what is the area coloured in the first stage? In the second stage? In the nth stage?

(iii) What is the fractal dimension of the resulting fractal?

(b) (10 marks) The hexagonal Game of Life is a variant of the Game of Life, played on a hexagonal grid, so that each cell has six neighbours. The rules are the same, namely:

- A dead cell comes alive if it has exactly 3 live neighbours.
- A live cell remains alive if it has 2 or 3 live neighbours.
- A live cell dies if it has fewer than 2 or more than 3 live neighbours.

(i) Using these rules, find the next two generations from the following starting configuration (grey cells are alive, white cells are dead). You should answer using the grids on the special yellow answer sheet provided.



(ii) Find a stable configuration with *at least* 4 live cells. Use the grid on the special yellow answer sheet provided to show your answer.

5. (a) (6 marks) Consider the following letters and shapes:

$$\alpha, \mathsf{B}, \delta, \Delta, \epsilon, \theta, \lambda, \nu, \pi, \rho, \tau, \sigma, \varphi, \chi, \psi, \Xi, \infty$$

Arrange these shapes into groups that consist of shapes that are equivalent to one another by distortion.

(b) (6 marks) Is a sphere equivalent by distortion to a torus? Explain your answer carefully.

(c) (8 marks) Suppose that for some given connected graph in the plane we have

$$V - E + F = 2$$

where V is the number of vertices, E is the number of edges, and F is the number of regions.

(i) Explain why the same relation will hold if we add one more edge.

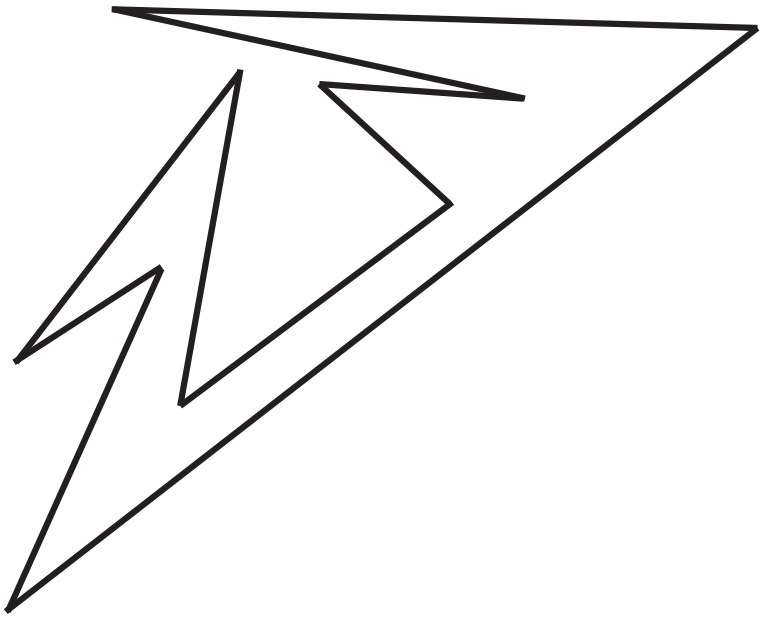
(ii) Can we conclude the relation holds for any finite connected graph in the plane?

Use a diagram to illustrate your answers.

Candidate's Name: _____ ID No: _____

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

Answer sheet for Question 3



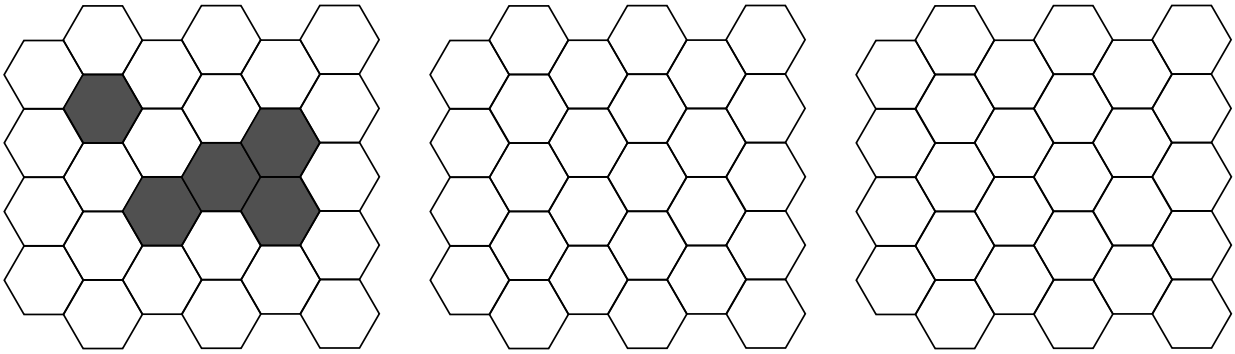
ATTACHMENT FOLLOWS

Candidate's Name: _____ ID No: _____

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

Answer sheet for Question 4

(b)(i)



(b)(ii)