THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2010
Campus: City

MATHEMATICS

Great Ideas Shaping our World

(Time allowed: TWO hours)

NOTE: This paper contains **9** questions. Answer **ALL** questions. Show **ALL** your working. There are 100 marks available in total.

1. (6 marks) A bank allocates 9-digit account numbers using a check digit system. If the account number is $n_1n_2n_3n_4n_5n_6n_7n_8n_9$ then the number

$$7n_1 + 3n_2 + 9n_3 + 7n_4 + 3n_5 + 9n_6 + 7n_7 + 3n_8 + 9n_9$$

is equal to 0 mod 10. For each of the following bank account numbers, one digit has been covered by a black square. Calculate what the correct digit should be. Show all your working.

(a) 21187294■

(b) 1■3288019

2. (10 marks) This question is about the Fibonacci sequence.

(a) Describe, either in words or using a formula, how the Fibonacci sequence is formed and write down the first 10 numbers in the sequence.

(b) The sequence of numbers P_n is defined so that P_n is the sum of the first n Fibonacci numbers. That is, $P_n = F_1 + F_2 + \cdots + F_n$. Using your answer to (a), write down the first 10 numbers of the sequence P_n.

(c) Compare your sequences F_n and P_n and find a simple formula which gives P_n in terms of one or more of the numbers in the Fibonacci sequence (other than the one given in (b)).

3. (10 marks)

(a) What is a prime number?

(b) Give two examples of numbers which are prime, and two examples of numbers which are not prime.

(c) Suppose you wanted to find out whether or not the number 93563 was prime. Describe (in two or three sentences) **how** you would do this. *You do not have to do any calculations for this part of the question.*

4. (15 marks) Let N be the set of natural numbers, i.e.,

$$N = \{1, 2, 3, 4, \dots\}.$$

For each of the following sets, state whether the set has cardinality that is less than the cardinality of N or equal to the cardinality of N or greater than the cardinality of N.

In each case, prove that the cardinality of the set is as you claim.

(a) The set O of odd integers

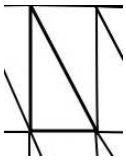
$$O = \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$$

(b) The set T of numbers between 0 and 1 for which the decimal expansion contains only the digits 2 and 3 after the decimal point. For example, $0.22233323\dots$ may be in T but $0.2223334323\dots$ is not.

(c) The set R of all rational numbers between 0 and 1.

5. (15 marks) Consider the following tile (it is a right-angle triangle with sides of length 1, 2 and $\sqrt{5}$).

(a) One can tile the plane using this tile as follows:



Briefly explain why this tiling can be extended to cover the whole plane.

(b) Which types of rigid symmetry (i.e., shifts, rotations and reflections) does this tiling have? Be precise (e.g., rotations of which angle and around which point) and briefly explain your answer.

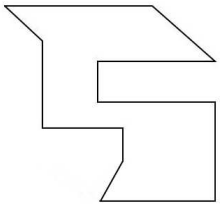
(c) Show that the tiling has symmetry of scale. Draw a supertile for this tiling.

(d) Briefly explain how to obtain a different tiling of the plane, using this same tile, which has no rigid symmetries but does have symmetry of scale.

6. (10 marks)

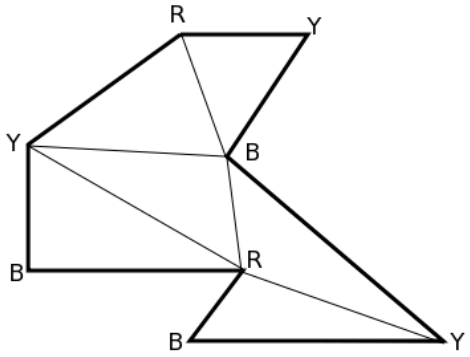
(a) The following picture is the floor plan of an art gallery. What is the smallest number of guards needed so that every wall can be viewed by a guard? (Remember that guards are only allowed to stand at corners.)

Use the copy of the figure provided on the answer sheet attached to the question paper to clearly mark the positions of the guards for your solution.



(b) State the art gallery theorem.

(c) The art gallery theorem is proved as follows: First one triangulates the gallery by adding straight lines between existing vertices (i.e., corners) of the art gallery. Then one colours the vertices using three colours (say, Red, Yellow and Blue), so that the three vertices of every triangle have different colours. The picture below shows an example of this stage of the proof.



Explain the remaining steps of the proof of the art gallery theorem (for the general case, not just for this picture). In particular, answer the questions:

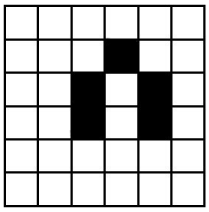
(i) Where should guards be put?

(ii) Why is it certain that the guards can see all the walls?

(iii) Why does this give an upper bound on the number of guards needed?

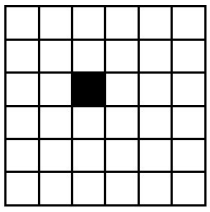
7. (9 marks) The answer sheet attached to the back of the question paper contains grids for your answers to this question.

(a) Consider the following initial population in the game of life.



Draw, on the grids on the answer sheet, the population in the next two steps.

(b) Find a population in the game of life whose next step is the following picture. Draw your solution on the grid on the answer sheet.



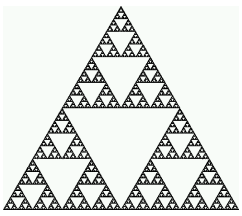
8. (10 marks)

(a) Briefly explain the meaning of the phrase "self-similarity". Give an example of an object in the real world which appears to have self-similarity.

(b) Consider the fractal obtained by repeating the following process: Take a solid line (of length 1) and delete the middle third (see picture below). Then repeat the process on each remaining line segment infinitely many times

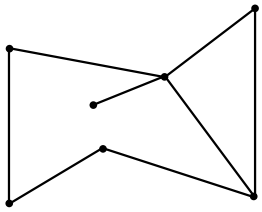
Draw the next two steps in the construction of this fractal.

(c) What is the fractal dimension of the following shape?



9. (15 marks)

(a) How many vertices and edges does the following graph have?



(b) Show that this graph cannot have an Euler circuit.

(c) What is the minimum number of edges that need to be added to this graph to obtain a graph which has an Euler circuit? Use the copy of the figure on the answer sheet attached to the question paper to draw the added edge(s) to the picture.

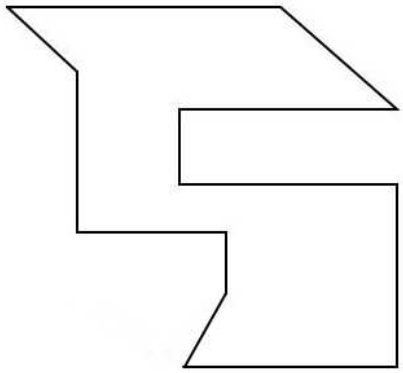
(d) Write down Euler's formula relating the number of vertices, edges and faces of a graph. How many faces does the graph in part (a) have? Verify Euler's formula in this case.

(e) You will now prove a special case of Euler's formula: Suppose one has a graph which satisfies Euler's formula. Now suppose one adds a single edge between two existing vertices to get a new graph. Explain why the new graph satisfies Euler's formula.

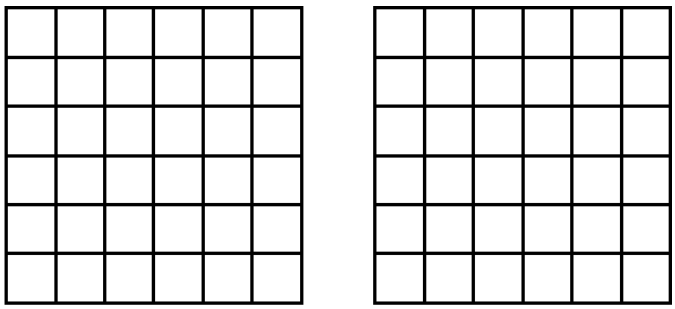
Candidate's Name: _____ ID No: _____

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

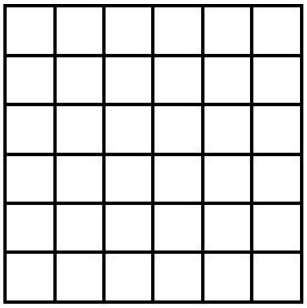
Answer sheet for Question 6(a):



Answer sheet for Question 7(a):



Answer sheet for Question 7(b):



Answer sheet for Question 9(b):