DEPARTMENT OF MATHEMATICS

MATHS 190 **Tutorial 4**

Tutorials in Maths 190 are **collaborative tutorials**. You should work in groups of 3 or 4 students, discussing the situations and puzzles listed below, or issues arising from lectures. Part of your final mark depends on your participation in tutorials.

A written answer to question 7 should be handed in for marking with your answers to Assignment 2 (due August 25th).

1. Your first job every morning at tennis camp is to get the ball machine ready for action. You open up some new cans of tennis balls and empty them into a large hopper. If each can contains three balls, is there a one-to-one correspondence between the balls and the cans?

2. We often use a method of checking whether two finite sets of objects have the same numbers of things by pairing and removing one object from each set until no objects remain. If we run out of objects from both sets at the same time, then we know the sets contain the same number of things. Otherwise, we know that one set is larger than the other. Describe several scenarios where we compare the size of two collections without computing individual sizes - for example, we can tell whether or not there is a chocolate biscuit missing from a packet by whether there are any empty spaces in the biscuit tray.

3. We can denote the natural numbers symbolically as $\{1, 2, 3, 4, \ldots\}$. Use this notation to express each of the sets described below.

    - The set of natural numbers less than 10.
    - The set of all even natural numbers.
    - The set of solutions to the equation $x^2 - 4 = 0$.
    - The set of all reciprocals of the natural numbers.

4. Let EIF be the set of all natural numbers ending in 5, i.e.,

    $$\text{EIF} = \{5, 15, 25, 35, 45, 55, 65, \ldots\}$$

    Describe a one-to-one correspondence between the set of natural numbers and EIF.

5. Let TIM be the set of all natural numbers except the number 3, so

    $$\text{TIM} = \{1, 2, 4, 5, 6, 7, 8, 9, \ldots\}$$

    Show that the set TIM and the set of all natural numbers have the same cardinality by describing an explicit one-to-one correspondence between the two sets.

6. (a) Suppose you have a finite number of pigeons and a finite number of holes. You try a method of assigning pigeons to holes (one pigeon to each hole) and, after filling all the holes, some pigeons remain. If you remove the pigeons and try again, is there any hope of placing each pigeon in an individual hole the second time?

   (b) Suppose now you have an infinite number of pigeons and holes. Is it possible that a first attempt to give each pigeon an individual hole fails but a second attempt succeeds?

7. **Write up your answer to this question and hand it in with your answers to Assignment 2 (due August 25th).** Don't forget to write down the names of the people in your tutorial group, so that you can acknowledge your collaborators in your report.

   Suppose you change the rules of the Ping-Pong Ball experiment discussed in lectures. This time you dump into the barrel 10 Ping-Pong balls numbered 1–10 as before and remove number 1. But next you put in 100 Ping-Pong balls, numbered 11–110, and remove number 2. Then you put in 1000 balls, numbered 111–1110 and remove number 3, and so on. The question is: How many Ping-Pong balls remain in the barrel after the stopwatch beeps? Infinitely many? Finitely many? Can you name one? Explain your answer clearly.

8. It is the stranded traveller's fantasy. The Hotel Cardinality is a full-service luxury hotel with bar and restaurant. It has as many rooms as there are natural numbers. The room numbers are $1, 2, 3, 4, 5, \ldots$. You can see why stranded travellers love the Hotel Cardinality. There appears to be no need for the sad sign: No Vacancy. Suppose, however, that every room is occupied.

   (a) If a weary traveller were to arrive late at night looking for a place to stay, could the manager figure out a way to provide the traveller with a private room (no sharing) without evicting another guest? The answer is yes. Describe how this can be done; of course, some guests will have to move to other rooms.

   (b) Now suppose that two more travellers arrive, each wanting his or her own private room. Is it possible for the manager to make room for these folks without pushing anyone onto the streets?

   (c) Suppose that now the hotel is full, the Infinite Life Insurance Company, which has lots of employees (in fact, there are as many employees as there are natural numbers) decides to provide each of its employees with a private room. Is it possible for the manager to give each of the infinitely many employees his or her own room without kicking any of the other guests onto the streets? (Note that the employees travel together on an Infinite Bus, which has enough seats so each employee has his/her own seat.)

   (d) **Harder:** The Infinite Bus Company built one bus for every room in the hotel. The hotel is full, and all the Infinite busses turn up at once, each full of passengers. Can the manager of the hotel give each bus passenger a room without kicking out any of the other guests?