Maths 761 Lecture 22

Topic for today
Homoclinic bifurcations of a saddle-focus

Reading for this lecture
Glendinning §12.4

Reading for next lecture
Glendinning §12.4

Today's handouts
Lecture 22 notes
Lab 11
Assignment 4

Homoclinic bifurcations of a saddle-focus

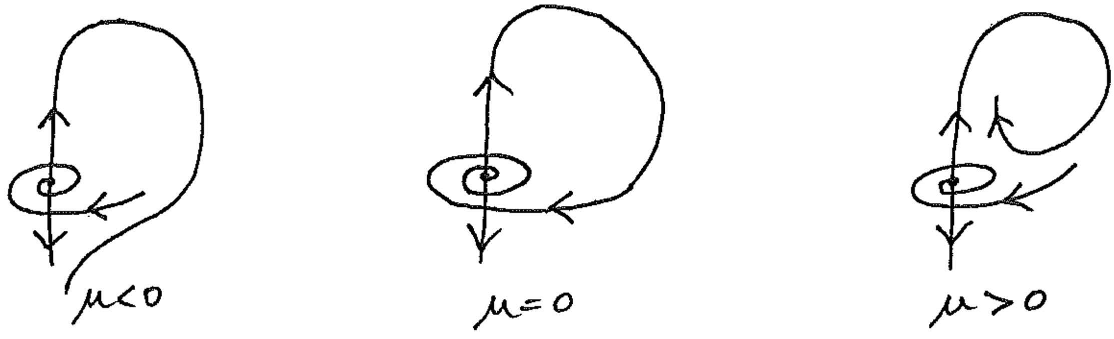
An equilibrium in $\mathbf{R}^3$ with eigenvalues $\lambda > 0$ and $\gamma = -\rho + i\omega$ with $\rho, \omega > 0$ is called a *saddle-focus*. We consider the bifurcations associated with the formation of a homoclinic orbit of a saddle-focus in $\mathbf{R}^3$.

Assume that the saddle-focus is at the origin, and that coordinates are chosen so that the linearised flow at the origin is

$$\begin{aligned}\dot{x} &= -\rho x - \omega y \\ \dot{y} &= \omega x - \rho y \\ \dot{z} &= \lambda z\end{aligned}$$

Then the stable manifold of the origin is tangent to the $x-y$ plane and the unstable manifold is tangent to the z-axis.

Assume that a homoclinic orbit exists at $\mu = 0$ and the orientation of the stable and unstable manifolds of the origin for $\mu \approx 0$ is as in the figure below.



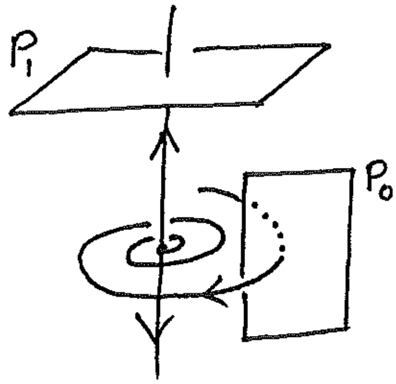
We construct a geometric return map to model the behaviour of solutions near the homoclinic orbit. We focus on the existence and stability of periodic solutions.

Near the origin, define two cross-sections:

$$P_0 = \{(x,y,z) \mid y = 0, |z| < \epsilon,\ \epsilon e^{-2\pi\rho/\omega} < x < \epsilon\}$$

and

$$P_1 = \{(x,y,z) \mid z = \epsilon, -\epsilon < x, y < \epsilon\}.$$



Then the flow induces two maps: a *local* map $\tilde{\Sigma}_0 : P_0 \to P_1$, and a *global* map $\tilde{\Sigma}_1 : P_1 \to P_0$. We derive approximate local and global maps Σ_0 and Σ_1 valid near the homoclinic orbit. Composition of these maps gives a return map which can be analyzed to give information about periodic orbits in the flow near the homoclinic orbit.

An approximate local map, Σ_0 is obtained using the linearised flow:

$$\begin{aligned} \dot{x} &= -\rho x - \omega y \\ \dot{y} &= \omega x - \rho y \\ \dot{z} &= \lambda z \end{aligned}$$

After some algebra, we find

$$\Sigma_0(x, 0, z) = (X, Y, \epsilon)$$

where

$$\begin{aligned} X &= x\left(\frac{z}{\epsilon}\right)^{\rho/\lambda} \cos\left\{-\frac{\omega}{\lambda}\log\left(\frac{z}{\epsilon}\right)\right\} \\ Y &= x\left(\frac{z}{\epsilon}\right)^{\rho/\lambda} \sin\left\{-\frac{\omega}{\lambda}\log\left(\frac{z}{\epsilon}\right)\right\} \end{aligned}$$

An approximate global map, Σ_1 is obtained using Taylor expansion about the heteroclinic connection that exists at $\mu = 0$. After some work, we find

$$\Sigma_1(X, Y, \epsilon) = (x', 0, z')$$

where

$$\begin{aligned} x' &= x^* + a\mu + cX + dY + \dots \\ z' &= \qquad\ \ b\mu + fX + gY + \dots \end{aligned}$$

with $b > 0$.

Composition of the local and global maps gives an approximate return map:

$$\Sigma = \Sigma_1 \circ \Sigma_0 : P_0 \to P_0$$

where

$$\Sigma(x, 0, z) = (x', 0, z')$$

and

$$\begin{aligned} x' &= x^* + a\mu + \alpha x z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_1\right\} + \dots \\ z' &= \quad + b\mu + \beta x z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_2\right\} + \dots \end{aligned}$$

where a, b, α, β, Φ_1 and Φ_2 are constants, and $\delta = \rho/\lambda$ is the ratio of the real parts of the eigenvalues.

This map is valid only for sufficiently small z and μ, and is defined for $z > 0$ only. We are interested in the existence and stability of fixed points for this map. Fixed points of the map correspond to periodic orbits of the flow that pass near the equilibrium once each period. Periodic orbits of the map of period k correspond to periodic orbits of the flow that pass near the equilibrium k times in each period.

The case $\delta > 1$ The return map is

$$\Sigma(x, 0, z) = (x', 0, z')$$

with

$$\begin{aligned} x' &= x^* + a\mu + \alpha x z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_1\right\} + \dots \\ z' &= \quad + b\mu + \beta x z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_2\right\} + \dots \end{aligned}$$

(Since $\delta > 1$ and z small, we can ignore some of these terms.) Fixed points of the map then satisfy

$$\begin{aligned} x &= x^* + O(\mu, x z^\delta) \\ z &= b\mu + O(\mu^2, x z^\delta) \end{aligned}$$

Since $b > 0$, there is a fixed point at $(x, z) \approx (x^*, b\mu)$ if $\mu > 0$, and no fixed point if $\mu < 0$. It is straightforward (and tedious) to compute the Jacobian for the map evaluated at the fixed point. It can be shown that the trace of the Jacobian is $O(z^{\delta-1})$ and the determinant is $O(z^{2\delta-1})$.

The variation of the period of the orbit as the bifurcation is approached can be calculated as for the planar case. The period is dominated by the time spent "in the box" near the equilibrium. The time in the box is

$$T = -\frac{1}{\lambda}\log z$$

where z is the height at which the periodic orbit intersects P_0. As $\mu \to 0$, the z value for the orbit goes to zero according to $z = b\mu$. Hence, the period of the orbit is

$$T \sim -\frac{1}{\lambda} \log \mu.$$

Thus the case $\delta > 1$ gives essentially the same bifurcation as the planar homoclinic case: the bifurcation adds one stable periodic orbit to the dynamics, with the period of the orbit increasing monotonically (growing as log of the bifurcation parameter) as the bifurcation is approached.