The Spectral Theorem for Bounded, Unbounded Hermitian and Normal Operators

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Abstract

In a Hilbert space $\mathcal{H}$ and measure space $L_2(X)$, the spectral theorem provides a representation of all normal operators $\mathcal{N}$ in $\mathcal{H}$, such that $\mathcal{N} = UMU^{-1}$ where U is a unitary operator in from $L_2(X) \mapsto \mathcal{H}$, U^{-1} its inverse, and M an operator on $L_2(X)$ representing a multiplication.

For $\mathcal{H}$ finite, this can be proved using various theorems involving the decomposition of matrices in terms of eigenvectors and the Schur decomposition (at least, where the underlying field is complex). However this does not generalise well into the infinite dimensional context.

The focus will be on a tractible proof applicable to when $\mathcal{H}$ is potentially infinite. Using Halmos' construction, the bounded Hermitian case can be proved in terms of elementary techniques, making it relatively accessible. It only requires familiarity with some elementary measure theory involving Riesz's representation theorem for positive linear functionals, functional analysis (Gelfand-Neimark theorems), Stone-Weierstraß theorem, and an argument involving Zorn's lemma.

Some modifications to this argument will yield the bounded normal operator case, and other forms of the spectral theorem including the spectral measure version.

By developing further definitions and machinery regarding unbounded operators, the spectral theorem can be proved for the unbounded Hermitian and normal operator cases by making use of the theorem in the bounded operator case. Only the unbounded Hermitian case will be presented.

As an application, the Fourier transform under some special L_2 spaces is shown to be representable by the spectral theorem.