DEPARTMENT OF MATHEMATICS
MATHS 761 Worksheet 6 - An example with local bifurcations

This worksheet shows you how to determine the bifurcations that occur in the system of equations:

$$\begin{aligned}\dot{x} &= \lambda x + 2xy + xy^2,\\ \dot{y} &= 1 - x^2 - y^2,\end{aligned}$$

as λ is varied.

Step 1 : Find all stationary solutions. You should find four. For each stationary solution, determine the range of λ for which it exists.

Step 2 : Determine the stability of each stationary solution over the range of λ values for which it exists. For two of the solutions this is easy - just calculate the eigenvalues directly. For the other solutions you need to be tricky, as follows. Call the stationary solutions (x_0, y_0) and (x_1, y_1). Then determine $\text{tr}(Df)$ and $\det(Df)$ in terms of x_0, y_0 or x_1, y_1. Then calculate the values of μ for which the quantities $\text{tr}(Df)$ and $\det(Df)$ are zero. Pitchfork, saddle-node or transcritical bifurcations occur when $\det(Df) = 0$ while a Hopf bifurcation occurs if $\text{tr}(Df) = 0$ and $\det(Df) > 0$. You should now be able to say where all the bifurcations of fixed points occur, although you probably won't be able to say where (x_0, y_0) or (x_1, y_1) changes from a node to a spiral.

Step 3 : Draw a bifurcation diagram (try plotting y versus λ). Make sure it is self-consistent, i.e., that solutions change stability only at bifurcations you identified in Step 2 and that you understand.

Step 4 : What happens to the periodic orbit that appears in the Hopf bifurcation? To answer this, it may help to draw phase portraits for various values of λ and compare them.

What is the stability of the periodic orbit that appears in the Hopf bifurcation? How do you know?