Maths 761 Lecture 14

Topic for today
Extended centre manifolds

Reading for this lecture
Glendinning Chapter 8, §8.2, §8.6
Strogatz Chapter 3 (all), §8.0, §8.1

Suggested exercises
Worksheet 5

Reading for next lecture
Glendinning §8.3-§8.6

Today's handouts
Lecture 14 notes
Worksheet 5

Extended Centre Manifolds

Centre manifold theory can be used to determine the behaviour of those solutions to a system of differential equations that lie close in phase space to a nonhyperbolic stationary solution. If we adapt the methods of the last lecture slightly, we may also be able to use centre manifold theory to determine the behaviour of solutions for parameter values close (in the space of parameters) to a bifurcation value.

Consider the system of equations

$$\dot{x} = f(x; \mu)$$

where $x \in \mathbf{R}^n$, $\mu \in \mathbf{R}^m$. Suppose that when $\mu = 0$, $x = 0$ is a nonhyperbolic stationary point for this system. Following the CM theorem, there exist coordinates

$$(u, v, w) \in E^c \times E^s \times E^u$$

such that at $\mu = 0$ the system can be written as

$$\begin{aligned} \dot{u} &= Au + f_1(u, v, w), \\ \dot{v} &= -Bv + f_2(u, v, w), \\ \dot{w} &= Cw + f_3(u, v, w), \end{aligned}$$

where A, B and C are matrices, all the eigenvalues of A have zero real part, all the eigenvalues of B and C have positive real part, and the functions f_i have no constant or linear terms.

If μ is now allowed to vary near zero, the system can be written as

$$\begin{aligned} \dot{u} &= A(\mu)u + f_1(u, v, w; \mu), \\ \dot{v} &= -B(\mu)v + f_2(u, v, w; \mu), \\ \dot{w} &= C(\mu)w + f_3(u, v, w; \mu), \end{aligned}$$

where all the eigenvalues of $A(0)$ have zero real part, and all the eigenvalues of $B(\mu)$ and $C(\mu)$ have positive real part. Here we use the observation that if $B(\mu)$ and $C(\mu)$ are finite-dimensional matrices, the eigenvalues of $B(0)$ and $C(0)$ have positive real parts which are bounded away from zero, and hence the eigenvalues of $B(\mu)$ and $C(\mu)$ will have strictly positive real part if μ is varied close to zero.

We now look at the extended system of equations

$$
\begin{aligned}
\dot{u} &= A(\mu)u + f_1(u, v, w; \mu), \\
\dot{v} &= -B(\mu)v + f_2(u, v, w; \mu), \\
\dot{w} &= C(\mu)w + f_3(u, v, w; \mu), \\
\dot{\mu} &= 0.
\end{aligned}
$$

The CM of this extended system has dimension equal to the dimension of the original CM plus the dimension of the vector μ. We can calculate the dynamics on the extended CM using power series, exactly as we did for unparametrised systems in the last lecture. In this case, however, the results will be valid for small $\|u\|$ and small $\|\mu\|$ and we will be able to work out the behaviour of solutions near $u = 0$ for all μ near the bifurcation value $\mu = 0$.