Maths 761 Lecture 5 Summary

Topic for today

Existence of periodic orbits in the plane

Reading for this lecture

Glendinning: §5.5-§5.8
Strogatz: §6.8, §7.0-§7.3

Recommended problems

Glendinning: §5: 6, 7, 8, 10
Strogatz: 7.1.1-4, 7.3.1, 7.3.3, 7.3.5

Reading for next lecture

Glendinning: §5.1-§5.4
Strogatz: §6.1-§6.4

Existence of periodic orbits in the plane

It is sometimes possible to determine analytically whether or not a system of equations defined on the plane $\mathbb{R}^2$ has periodic solutions. In the following we consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

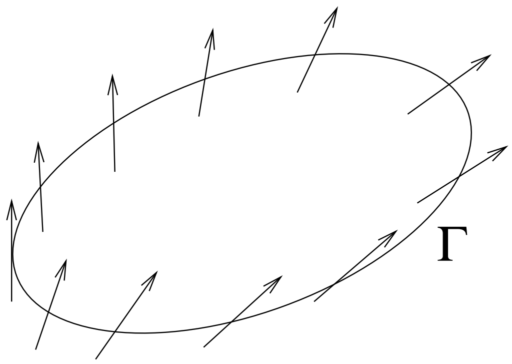
for $\mathbf{x} \in \mathbb{R}^2$, where $\mathbf{f}(x)$ is a C^1 function. The following results can be useful.

1. **Poincaré-Bendixson Theorem**
   Let $\mathbf{D}$ be a closed and bounded domain in the plane, and assume there are no stationary solutions of equations (1) in $\mathbf{D}$. If the trajectory through a point $\mathbf{x}_0$ enters and does not leave $\mathbf{D}$ (so $\phi(\mathbf{x}_0, t) \in \mathbf{D}$ for all $t \geq T$ for some $T \geq 0$) then there is at least one periodic orbit in $\mathbf{D}$ and this orbit is in the ω-limit set of $\mathbf{x}_0$. For a proof refer to Glendinning §5.8.

2. **The divergence test (a special case of Dulac's Criterion)**
   Let S be a simply connected subset of the plane. For system (1) above, if $\bigtriangledown \cdot \mathbf{f}$ has one sign throughout S then there are no closed orbits lying entirely in S. For a proof refer to Glendinning §5.6.

3. **Index Theory**

Let Γ be a closed curve in the phase plane that does not pass through any equilibria. This curve need not be a solution curve. At each point on Γ draw an arrow giving the slope and direction of the solution to (1) that passes through that point, as shown in the figure on the next page.



The arrows rotate as you move along Γ. In one complete *anticlockwise* trip about Γ the arrows will rotate through an angle $2\pi k$ where k is some integer. The *Poincaré index* of the curve Γ is defined to be this integer k.

If Γ encloses a single, isolated stationary solution, we define the index of the stationary solution to be the index of Γ. A sink or source or centre has index $+1$, while a saddle has an index of -1. A periodic orbit has index $+1$.

Theorem: The index of a closed curve (in the plane) is equal to the sum of the indices of the stationary solutions enclosed by the closed curve.

Corollary: Inside any closed orbit there must be at least one stationary solution. If there is only one, it must be a source or sink or centre. If all the stationary solutions inside the closed orbits are hyperbolic there must be an odd number, of which n are saddles and $n + 1$ are sinks or sources.