EXTRAPOLATION OF RUNGE KUTTA METHODS

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Abstract

Extrapolation [6] involves taking a certain linear combination of the numerical solutions of a base method applied with different stepsizes to obtain greater accuracy. This linear combination is done so as to eliminate the leading error term. Extrapolation can be applied in two different ways: 1) In the active mode the extrapolated value is used to propagate the numerical solution at each step; 2) in the passive mode extrapolation is carried out only at points where greater accuracy is desired but where the solutions of the base method are propagated without using the extrapolated value. The technique of extrapolation in accelerating convergence has been used successfully in the numerical solution of non-stiff ordinary differential equations (Gragg [4], Bulirsch and Stoer [2]). Gragg introduced the smoothing formula for the explicit midpoint rule while Bulirsch and Stoer developed the first code ODEX for the "Gragg-Bulirsch-Stoer algorithm" using the explicit midpoint rule as the base method. Meanwhile, the first successful extrapolation code developed for stiff problems is METAN1 by Bader and Deuflhard [1] which implements the linearly implicit midpoint rule.

In this study we investigate the implicit midpoint rule (IMR) denoted by $\mathcal{M}$ for stiff problems. For example, consider the problem $y'(x) = qy(x)$, $y(0) = 1$, where q is a large negative number. The exact solution is given by $y(x) = e^{qx}$ which decays rapidly when $|q|$ is large. However, if $|q|$ is large, the numerical solution, given by $y_n = \left(\frac{1+hq/2}{1-hq/2}\right)^n$, becomes oscillatory. The explicit midpoint rule for nonstiff problems contains a parasitic component that is also oscillatory. Gragg's smoothing formula dampens the oscillatory parasitic component. This smoothing formula has the same effect when applied to the IMR for stiff problems. For higher order symmetric methods a more complicated smoothing is required. Chan [3] generalized the concept of smoothing to arbitrary symmetric Runge-Kutta methods. Symmetric methods admit asymptotic error expansion in even powers of the stepsize and are therefore of special interest because successive extrapolations can increase the order by two at a time. The generalized smoothing is designed to preserve the h^2-asymptotic error expansion as well as to provide damping for stiff problems. Chan constructed L-stable methods called *symmetrizers* which posses these properties. The extrapolations applied to the symmetrized symmetric methods have much better behaviour than those applied without the symmetrizers. If $\tilde{\mathcal{R}}$ denotes a symmetrizer and $\mathcal{R}$ the symmetric method then the symmetrizer is applied at the very last step, $\frac{1}{n}(\mathcal{R}^{n-1} \circ \tilde{\mathcal{R}})$.

Results of numerical experiments with the Prothero-Robinson and Kaps problems are given which show the effects of smoothing on the IMR. The results suggest that higher order symmetric methods can also benefit from the use of symmetrizers. Hence, it will be interesting to determine the efficiency of these symmetrized methods of higher order for certain stiff problems and the effectiveness of these symmetrizers when applied to extrapolation in the passive and active modes.

References

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