Maths 761 Lecture 15

Topic for today
Local bifurcations: saddle-node, transcritical, and pitchfork bifurcations

Reading for this lecture
Glendinning Chapter 8, §8.3-§8.5

Reading for next lecture
Glendinning §8.8

Today's handout
Lecture 15 notes

Local Bifurcations

Suppose the system of equations $\dot{y} = g(y; \lambda)$ has a nonhyperbolic stationary point at $y = 0$ when $\lambda = 0$, where $y \in \mathbf{R}^k$ and $\lambda \in \mathbf{R}^l$. We can use extended centre manifold theory to simplify the problem of discovering how solutions behave near $y = 0$ when λ is near zero; instead of looking at the full k dimensional system of equations we can restrict attention to the $m \leq k$ nontrivial equations that describe the motion on the extended centre manifold. Here m is the dimension of the centre manifold in the original problem, i.e. m is the number of eigenvalues of $Dg(0; 0)$ that have zero real part.

It is usual to classify local bifurcations of stationary solutions according to the number of critical eigenvalues of the Jacobian (i.e. the number of eigenvalues with zero real part) at the bifurcation. The simplest, and most common, bifurcations are those with the smallest numbers of critical eigenvalues. We will study mainly the cases when there is one zero eigenvalue or one pair of purely imaginary eigenvalues at the bifurcation, and for now restrict attention to the case that there is only one parameter, μ.

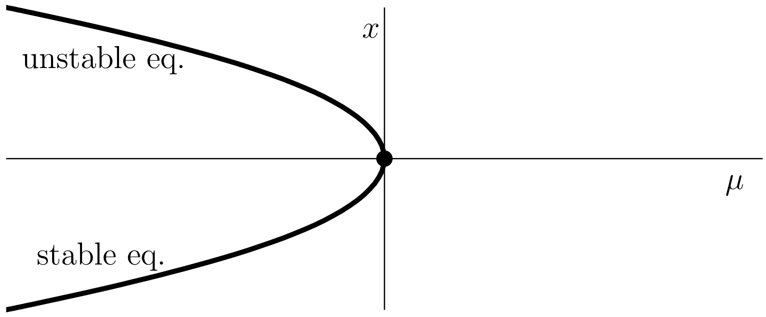
In the case of one zero eigenvalue, the dynamics on the centre manifold is determined by a single equation, $\dot{x} = f(x; \mu)$ where $x \in \mathbf{R}$. Expanding this equation as a Taylor series in x and μ gives

$$\dot{x} = f(0; 0) + f_x x + f_\mu \mu + \frac{1}{2}\left(f_{xx}x^2 + 2f_{x\mu}x\mu + f_{\mu\mu}\mu^2\right) + \ldots$$

where all the partial derivatives are evaluated at $x = \mu = 0$. Because $x = 0$ is a nonhyperbolic stationary solution when $\mu = 0$, $f(0; 0) = f_x(0; 0) = 0$, and the dynamics on the centre manifold is determined by the values taken by the higher partial derivatives of f. We look at the three most common cases.

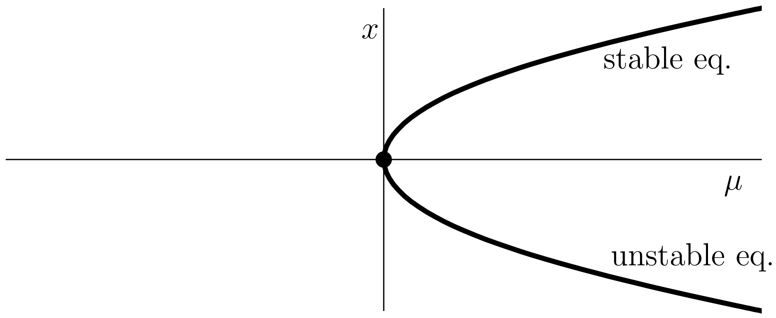
1. **The saddle-node bifurcation**

   Example 1: $\dot{x} = \mu + x^2$.
   This system has stationary solutions at $\pm\sqrt{-\mu}$ when $\mu < 0$. The Jacobian is $Df = 2x$, which implies that $x = \sqrt{-\mu}$ is an unstable solution and $x = -\sqrt{-\mu}$ is stable. The bifurcation diagram is:



   In this example, a pair of stationary solutions, of different stabilities, come together and annihilate one another as $\mu \to 0$ from below.

   Example 2: $\dot{x} = \mu - x^2$.
   This system has stationary solutions at $\pm\sqrt{\mu}$ when $\mu > 0$. The Jacobian is $Df = -2x$, so $x = \sqrt{\mu}$ is a stable solution and $x = -\sqrt{-\mu}$ is unstable. The bifurcation diagram is:



   In this example, a pair of stationary solutions, of different stabilities, bifurcate from $x = 0$ as μ increases from zero.

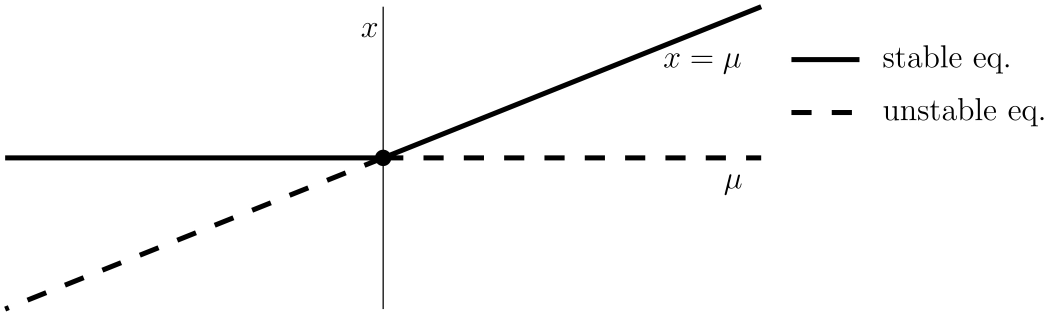
   More generally, we get

   The saddle-node bifurcation theorem
   Consider the equation $\dot{x} = f(x;\mu)$ where $x, \mu \in \mathbf{R}$ and $f(0;0) = f_x(0,0) = 0$.

   If $f_\mu(0;0) \neq 0$ and $f_{xx} \neq 0$ there is a continuous curve of stationary solutions in some neighbourhood of $(0,0)$ in the $x\mu$ plane; the curve is tangent to the line $\mu = 0$.

   If $f_\mu f_{xx} < 0$ (resp. $f_\mu f_{xx} > 0$) there are no stationary points near $(0,0)$ for $\mu < 0$ (resp. $\mu > 0$) while for each positive (resp. negative) value of μ sufficiently close to zero there are two stationary points with x values near zero. The stationary points are hyperbolic for $\mu \neq 0$ with the upper one being stable and the lower one unstable if $f_{xx} < 0$. The stabilities are reversed if $f_{xx} > 0$.

2. **The transcritical bifurcation**
   Example 3: $\dot{x} = \mu x - x^2$.
   This system has stationary solutions at $x = 0$ and $x = \mu$. The Jacobian is $Df = \mu - 2x$, so $x = 0$ is stable if $\mu < 0$ and unstable if $\mu > 0$, while $x = \mu$ is unstable if $\mu < 0$ and stable if $\mu > 0$. The bifurcation diagram is:



   In this example, a pair of stationary solutions, of different stabilities, come together, pass through one another, and separate again as μ passes through zero. The stationary solutions exchange stability at the bifurcation.

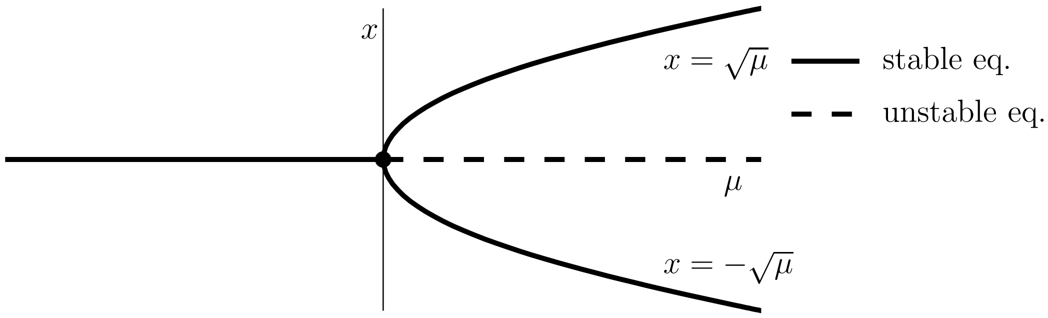
   More generally, we get

   The transcritical bifurcation theorem
   Consider the equation $\dot{x} = f(x;\mu)$ where $x, \mu \in \mathbf{R}$ and $f(0;0) = f_x(0,0) = 0$.

   If $f_\mu(0;0) = 0$, $f_{xx} \neq 0$, and $f_{\mu x}^2 - f_{xx}f_{\mu\mu} > 0$ there are two continuous curves of stationary solutions in some neighbourhood of $(0,0)$ in the $x\mu$ plane. The curves intersect transversally at $(0,0)$.

   For each sufficiently small, nonzero μ there are two hyperbolic stationary solutions near $x = 0$, with the upper one stable (resp. unstable) and the lower one unstable (resp. stable) if $f_{xx} < 0$ (resp. $f_{xx} > 0$).

3. **The pitchfork bifurcation**
   Example 4: $\dot{x} = \mu x - x^3$.
   This system has stationary solutions at $x = 0$ for all μ and at $x = \pm\sqrt{\mu}$ if $\mu > 0$. The Jacobian is $Df = \mu - 3x^2$, so $x = 0$ is stable if $\mu < 0$ and unstable if $\mu > 0$, while the solutions $x = \pm\mu$ are stable when they exist.



   In this example, a stationary stable solution becomes unstable at $\mu = 0$, throwing off a pair of stable stationary solutions at the bifurcation.

More generally, we get

The pitchfork bifurcation theorem
Consider the equation $\dot{x} = f(x;\mu)$ where $x, \mu \in \mathbf{R}$ and $f(0;0) = f_x(0,0) = 0$.

If $f_\mu(0;0) = f_{xx} = 0$, $f_{\mu x} \neq 0$ and $f_{xxx} \neq 0$, there are two continuous curves of stationary solutions in some neighbourhood of $(0,0)$ in the $x\mu$ plane. One of the curves passes through $(0,0)$ transverse to the axis $\mu = 0$ while the other is tangent to $\mu = 0$ at $(0,0)$.

If $f_{\mu x} f_{xxx} < 0$ there exist three stationary points near $x = 0$ if $\mu > 0$ (the outer pair being stable and the inner one unstable if $f_{xxx} < 0$), and one stationary point near $x = 0$ (stable if $f_{xxx} < 0$) if $\mu < 0$.

If $f_{\mu x} f_{xxx} > 0$ there exist three stationary points near $x = 0$ if $\mu < 0$ (the outer pair being stable and the inner one unstable if $f_{xxx} < 0$), and one stationary point near $x = 0$ (stable if $f_{xxx} < 0$) if $\mu > 0$. The stabilities are reversed if $f_{xxx} > 0$.