

Department of Mathematics

MATHS 190/190G

Great Ideas Shaping our World

Mid-semester Test

20 September 2010

Instructions

- This test contains 6 questions. Attempt **ALL** questions.
- Total marks = 50.
- Show **ALL** your working and explain all of your answers carefully.
- You have 1 hour to do the test.

1. (7 marks)

- (a) What is a prime number?
- (b) Let p_1 and p_2 be two prime numbers, and let $m = p_1p_2 + 1$.
 - i. What is the remainder when m is divided by p_1 ?
 - ii. Find two sets of examples for p_1 and p_2 , one where m is prime, and one where m is not prime.
- (c) Let p_1, p_2, \dots, p_{100} be the first 100 prime numbers, and let $n = p_1p_2 \dots p_{100} + 1$. What can you say about the divisors of n ? Is n necessarily prime?

2. (7 marks)

- (a) Write down the first 10 numbers in the Fibonacci sequence, F_n , where $F_1 = F_2 = 1$.
- (b) A sequence of numbers P_n is created using the rule $P_n = P_{n-1} + P_{n-2} + P_{n-3}$. That is, each number is the sum of the *three* previous terms in the sequence. The starting numbers are $P_1 = P_2 = P_3 = 1$.
 - i. Write down the first 10 numbers in the sequence.
 - ii. Use your answers to (a) and (b)i. to write down the values of n (for n between 1 and 10) for which $P_n > F_n$.
 - iii. Is P_n always greater than F_n for $n > 10$? Explain your answer carefully.

3. (8 marks)

A tube of Smarties contains 39 sweets, which can be any of 8 different colours. Claire and Ivo divide the contents of a tube of Smarties between themselves. What is the largest number you can write in place of the '?' in the statements below, and be sure they are correct, however the Smarties are divided? Include an explanation with each answer.

- (a) The tube will contain at least ? Smarties which are the same colour.
- (b) The tube will contain at least ? red Smarties.
- (c) At least one of Claire and Ivo will have at least ? Smarties.
- (d) Ivo will get at least ? blue Smarties.

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4. (12 marks)

- (a) Carefully explain what it means for a set to be 'infinite'.
- (b) Consider two *finite* sets S and T , which are too large to practicably count. How could you easily check which is the larger set?
- (c) Explain carefully why the set

$$\{\dots, -25, -16, -9, -4, -1, 1, 4, 9, 16, 25, \dots\}$$

has the same cardinality as the set of natural numbers $\{1, 2, 3, \dots\}$.

- (d) Use Cantor's diagonalisation process to show that the set of real numbers in the interval $[0, 1]$ has a greater cardinality than the set of natural numbers.

5. (8 marks) For each of the following questions, include a picture in your explanation.

- (a) Which interval has a larger cardinality, $[2, 6]$ or $[-1, 15]$?
- (b) Explain carefully why the set of points on a line segment has the same cardinality as the set of real numbers.

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6. (8 marks) Consider the floor plan of the art gallery shown in Figure 1. Guards must be placed at the corners of the gallery so that each (straight) wall can be seen by at least one guard. The guards can turn their heads but are not allowed to leave their corners.

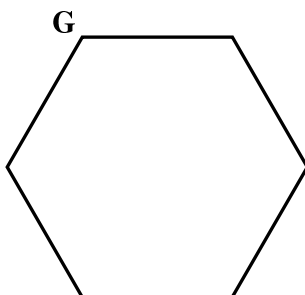


Figure 1: Art gallery shaped like a regular hexagon

- Suppose you are a guard standing at corner G . Show (with a picture) that you can see all other corners from G .
- What is the smallest number of guards that you need to guard the gallery shown in Figure 1?
- In general, what is the minimal number of guards needed if the gallery has a corner from which you can see all other corners? Explain your answer carefully.