Maths 761 Lecture 8 Summary

Topic for today

Stability of fixed points in linear maps
Linearisation in nonlinear maps

Reading for today's lecture

Glendinning: §6.1, §6.5

Recommended problems

Glendinning: §6: 3,4,5

Reading for next lecture

Glendinning: §11.3-5

Stability of Fixed Points in Linear Maps

Consider the linear difference equation $x_{n+1} = Ax_n$, where A is an $m \times m$ constant matrix. The origin $x = 0$ is a fixed point of this map. If λ is an eigenvalue of A with eigenvector v, then $x_n = \lambda^n v$ is a solution to the difference equation. In the case that all the eigenvalues of A are distinct, the general solution to the difference equation is:

$$x_n = c_1\lambda_1^n v_1 + c_2\lambda_2^n v_2 + ... + c_m\lambda_m^n v_m$$

where the eigenvalues are $\lambda_1, \lambda_2, ..., \lambda_m$ and the eigenvectors are $v_1, v_2, ..., v_m$. The case that there are repeated eigenvalues is more complicated - see Worksheet 4.

If all the eigenvalues λ_i of A have $|\lambda_i| < 1$ then all solutions of the difference equation tend to the origin as $n \to \infty$.

The stable, unstable and centre subspaces (or manifolds) for fixed points of linear maps are defined in an analogous way to the corresponding subspaces for flows. Thus, we define:

1. $E^u(0)$, the *unstable manifold (or subspace) of the origin*, is the span of the eigenvectors and generalised eigenvectors corresponding to the eigenvalues of A with modulus greater than one.

2. $E^s(0)$, the *stable manifold (or subspace) of the origin*, is the span of the eigenvectors and generalised eigenvectors corresponding to the eigenvalues of A with modulus less than one.

3. $E^c(0)$, the *centre manifold (or subspace) of the origin*, is the span of the eigenvectors and generalised eigenvectors corresponding to the eigenvalues of A with modulus equal to one.

Stability of Fixed Points in Nonlinear Maps

Consider the map

$$x_{j+1} = f(x_j)$$

with a fixed point at $x_j = x^\star$. A fixed point $x^\star$ of the map is *hyperbolic* if and only if $|\lambda_i| \neq 1$ for all eigenvalues λ_i of the matrix $Df(x^\star)$.

The fixed point is asymptotically stable if all $|\lambda_i| < 1$. If some $|\lambda_i| < 1$ and some $|\lambda_i| > 1$ the fixed point is a saddle. If all $|\lambda_i| > 1$, the fixed point is a source.