Maths 761 Lecture 11 Summary

Topic for today

Structural Stability

Reading for lectures 11 and 12

Glendinning: §4.3, §4.4

Recommended problems

Glendinning: §4: 9

Topological equivalence

Consider two systems of differential equations, $\dot{x} = f(x)$ and $\dot{y} = g(y)$, where f and g are C^1 vector fields. Denote by $\phi_f(x,t)$ and $\phi_g(y,t)$ the flows corresponding to these two systems.

The systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are *topologically equivalent* if there is a homeomorphism, h, (i.e., a continuous function h with a continuous inverse), which takes orbits of $\dot{x} = f(x)$ to orbits of $\dot{y} = g(y)$, preserving the sense of parametrization by time. This means that for any x and t_1 there is a t_2 such that

$$h\left(\phi_f(x,t_1)\right) = \phi_g(h(x),t_2)). \tag{1}$$

If h preserves parametrization by time, i.e., if $t_1 = t_2$ in (1), then the systems are *topologically conjugate*.

Instead of saying that the systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are topologically equivalent or conjugate, we sometimes say that the vector fields f and g are topologically equivalent or conjugate.

Homeomorphisms preserve some important qualitative features of flows. For instance, a homeomorphism will map a stationary solution in one flow to a stationary solution in another flow. Similarly, periodic orbits and homoclinic connections are preserved by homeomorphisms. The condition that the sense of orbits be preserved by a homeomorphism if topological equivalence is to hold ensures that the stability (but not necessarily type) of solutions is preserved.

Structural Stability

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is *structurally stable* if for all twice differentiable vector fields v there is an $\epsilon_0 > 0$ such that f is topologically equivalent to $f + \epsilon v$ for all $\epsilon \in (0, \epsilon_0)$.