DEPARTMENT OF MATHEMATICS
MATHS 761 - Calculating W^s_{loc} and W^u_{loc}.

This handout contains two examples of how to calculate series expansions for W^s_{loc} and W^u_{loc}.

Example 1

The system

$$\begin{aligned}\dot{x} &= 2x + y^2\\ \dot{y} &= -y\end{aligned}$$

has a stationary solution at $(0,0)$, with Jacobian

$$Df(0,0) = \begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix} \Rightarrow E^u(0,0) = \text{sp}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}, E^s(0,0) = \text{sp}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

Thus W^s_{loc} is a one-dimensional manifold tangent to the y-axis, and W^u_{loc} is a one-dimensional manifold tangent to the x-axis.

To find an approximation to W^s_{loc}

Write $x = g(y) = \sum_{n=0}^{\infty} a_n y^n$ (assuming W^s_{loc} has a power series expansion).
We know $a_0 = 0$ (since W^s_{loc} passes throught the origin)
and $a_1 = 0$ (since W^s_{loc} is tangent to the y-axis).
On W^s_{loc},

$$\begin{aligned}\dot{x} &= \frac{dg}{dy}\dot{y} \text{ (by chain rule)}\\ &= (2a_2 y + 3a_3 y^2 + \dots + n a_n y^{n-1} + \dots)(-y)\\ &= -2a_2 y^2 - 3a_3 y^3 - \dots - n a_n y^n + \dots.\end{aligned} \tag{1}$$

Also,

$$\begin{aligned}\dot{x} &= 2x + y^2 \text{ (from the original differential equation)}\\ &= 2(a_2 y^2 + a_3 y^3 + \dots) + y^2 \text{ on } W^s_{\text{loc}}\\ &= (2a_2 + 1)y^2 + 2a_3 y^3 + \dots + 2a_n y^n + \dots.\end{aligned} \tag{2}$$

Equating coefficients of powers of x in (1) and (2), our two expressions for $\dot{x}$, yield:

$$\begin{aligned}-2a_2 &= 2a_2 + 1 && \Rightarrow a_2 = -\frac{1}{4}\\ -3a_3 &= 2a_3 && \Rightarrow a_3 = 0\\ &\vdots\\ -na_n &= 2a_n && \Rightarrow a_n = 0, n \geq 4.\end{aligned}$$

So $W^s_{\text{loc}} = \left\{(x,y) | x = g(y) = -\frac{1}{4}y^2\right\}$.

To find an approximation to W^u_{loc}

W^u_{loc} is tangent to the x-axis. In fact, the x-axis is invariant under the dynamics since $\dot{y} = 0$ if $y = 0$. We see that the only one-dimensional manifold tangent to the x-axis is in fact the x-axis itself $\Rightarrow W^u_{\text{loc}}$ is the x-axis.

Local phase portrait

Example 2

The system

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= -y + x^2 \end{aligned}$$

has two equilibria, $(0,0)$ and $(-2,4)$. Here we find $W^s_{\text{loc}}(0,0)$ and $W^u_{\text{loc}}(0,0)$. (To find $W^s_{\text{loc}}((-2,4)$ and $W^u_{\text{loc}}(-2,4)$, it is necessary to make a coordinate change that moves the equilibrium to the origin, then proceed as below.)

$$Df(0,0) = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow E^u(0,0) = \text{sp}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, E^s(0,0) = \text{sp}\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\}$$

Seek power series expansions for W^u_{loc} and W^s_{loc}.

For W^s_{loc}

Write $y = h(x) = \sum a_n x^n$.
Know that $a_0 = 0$ (since W^s_{loc} passes through $(0,0)$)
and that $a_1 = -3$ (since W^s_{loc} tangent to $y = -3x$).
For now, put $a_0 = 0$, but leave a_1 undetermined.

Then

$$
\begin{aligned}
\dot{y} &= \frac{dh}{dx}\dot{x} \\
&= \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)\left(2x + \sum_{n=1}^{\infty} a_n x^n\right) \\
&= 2\sum_{n=1}^{\infty} n a_n x^n + \left(\sum_{n=1}^{\infty} a_n x^n\right)\left(\sum_{n=1}^{\infty} m a_m x^{m-1}\right) \\
&= (2a_1 x + 4a_2 x^2 + \dots) + (a_1 x + a_2 x^2 + \dots)(a_1 + 2a_2 x + \dots) \\
&= x(2a_1 + a_1^2) + x^2(4a_2 + 2a_1 a_2 + a_1 a_2) + \dots
\end{aligned} \tag{3}
$$

Also,

$$
\begin{aligned}
\dot{y} &= -y + x^2 \\
&= -\sum_{n=1}^{\infty} a_n x^n + x^2 \\
&= -a_1 x + (1 - a_2)x^2 + \dots
\end{aligned} \tag{4}
$$

Equating coefficients of powers of x in (3) and (4):

$$
\begin{aligned}
-a_1 &= 2a_1 + a_1^2 \qquad \Rightarrow a_1 = 0 \text{ or } a_1 = -3 \\
1 - a_2 &= 4a_2 + 3a_1 a_2
\end{aligned}
$$

For now use $a_1 = -3$ (this is the correct choice for W^s_{loc}). Substitute into second equation:

$$
1 - a_2 = 4a_2 - 9a_2 \Rightarrow a_2 = -\frac{1}{4}
$$

So, to quadratic order, $W^s_{\text{loc}}(0,0) = \left\{(x,y) | y = -3x - \frac{1}{4}x^2\right\}$.

For W^u_{loc}

This is tangent to $y = 0$.
Write $y = h(x) = \sum a_n x^n$ as in calculations for W^s_{loc}. However, this time $a_1 = 0 \Rightarrow a_2 = \frac{1}{5}$ from working above. So $W^u_{\text{loc}}(0,0) = \left\{(x,y) | y = \frac{1}{5}x^2 + \text{ higher order terms } \right\}$.
Local phase portrait: