Maths 761 Lecture 13

Topics for today
Introduction to bifurcations
Bifurcations of stationary points: centre manifolds

Reading for this lecture
Glendinning Chapter 8 to end of §8.1

Suggested exercises
Glendinning Exercises 8, Numbers 1, 2

Reading for next lecture
Glendinning §8.2

Today's handouts
Lecture 13 notes
Handout: "Determining the dynamics near a non-hyperbolic ...

Introduction to bifurcations

We now consider parametrised families of systems of autonomous equations,

$$\dot{x} = f(x;\mu), \quad x \in \mathbf{R}^n, \quad \mu \in \mathbf{R}^m.$$

A **bifurcation** occurs when a change in the vector of parameters μ leads to a qualitative change in the dynamics. A value of the parameter, $\mu = \mu_0$, at which a bifurcation occurs is called a **bifurcation value**. Note that two systems of equations $\dot{x} = f(x;\mu_0)$ and $\dot{y} = g(y;\nu_0)$ are said to have qualitatively the same dynamics if they are topologically equivalent.

Bifurcations are commonly divided in two classes. A **local bifurcation** occurs when there is a qualitative change in the dynamics near a stationary point or periodic orbit, A **global bifurcation** occurs when global properties of the dynamics change qualitatively. The next several lectures are concerned with local bifurcations.

Local bifurcations

The number and/or stability of stationary or periodic solutions usually changes at a local bifurcation.

Example: Determine the dynamics associated with

$$\dot{x} = \mu - x^2, \quad x \in \mathbf{R}, \ \mu \in \mathbf{R}$$

for all values of μ.

Local bifurcations occur at parameter values for which a stationary or periodic solution is non-hyperbolic. Thus to understand local bifurcations we need to analyze the dynamics near a nonhyperbolic stationary or periodic solution. **Centre manifold theory** is useful for this.

The Centre Manifold Theorem

Consider the system of equations $\dot{x} = f(x)$ where $f \in C^r$, $f : \mathbf{R}^n \to \mathbf{R}^n$, and $f(0) = 0$. Let E^u, E^s, and E^c be the generalised eigenspaces associated with the eigenvalues of $Df(0)$ with negative, positive, and zero real part, respectively.

Then there exist C^r unstable and stable manifolds, $W^u(0)$ and $W^s(0)$ tangent to E^u and E^s, respectively, at $x = 0$, and of the same dimension as E^u and E^s, respectively. There is also a C^{r-1} manifold, $W^c(0)$, tangent to E^c at $x = 0$, and of the same dimension as $E^c(0)$. All three manifolds are invariant but W^c is not necessarily unique.

Important consequence of the centre manifold theorem:

Let $x = 0$ be a nonhyperbolic stationary point of the system $\dot{x} = f(x)$. The CM Theorem allows us to choose coordinates on the invariant subspaces:

$$(u, v, w) \in W^c \times W^s \times W^u$$

$$\begin{aligned} \dot{u} &= g(u), \\ \dot{v} &= -Bv, \\ \dot{w} &= Cw, \end{aligned}$$

or, if

$$(x, y, z) \in E^c \times E^s \times E^u,$$

as

$$\begin{aligned} \dot{x} &= A(x) + h_1(x), \\ \dot{y} &= -By + h_2(x, y, z), \\ \dot{z} &= Cz + h_3(x, y, z), \end{aligned}$$

where g, h_1, h_2 and h_3 are real-valued functions with the h_i having no constant or linear terms, and A, B and C are matrices with the eigenvalues of A having zero real part, and the eigenvalues of B and C having positive real part.

Because motion on the stable manifold is always towards the stationary point and motion on the unstable manifold is always away from the stationary point, we can determine the dynamics near the stationary point if we work out what is happening on the centre manifold.

The dynamics on the centre manifold is determined by calculating an approximation to the function g (using power series expansions) and analyzing the equation $\dot{u} = g(u)$.