Maths 761 Lecture 23

Topic for today
More on homoclinic bifurcations of saddle-foci

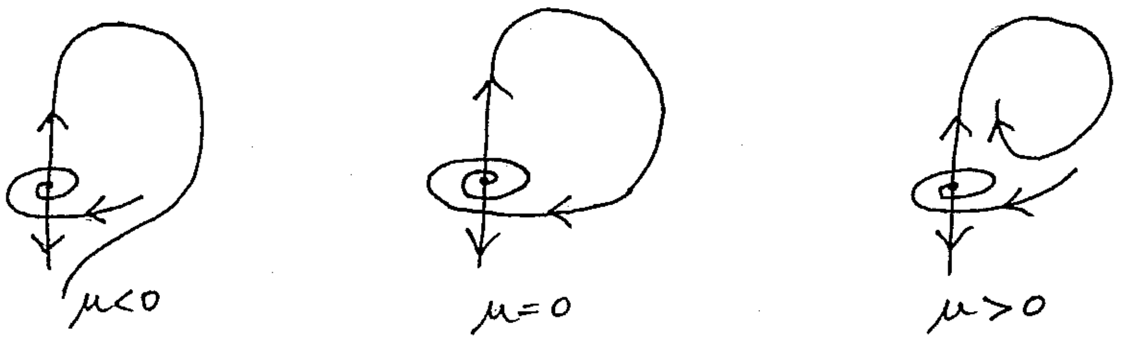
Reading for this lecture
Glendinning §12.4

Today's handouts
Lecture 23 notes

More on homoclinic bifurcations of saddle-foci

In the last lecture, we derived a return map that approximated the dynamics near a homoclinic bifurcation of a saddle-focus. Our main assumptions were:

- the saddle-focus is at the origin
- eigenvalues of the linearised flow at the origin are $-\rho \pm i\omega$, λ, with coordinates chosen so that the corresponding eigenspaces align with the coordinate axes
- a homoclinic orbit exists at $\mu = 0$ and the orientation of the stable and unstable manifolds of the origin for $\mu \approx 0$ is as shown below.



We constructed a geometric return map to model the behaviour of solutions near the homoclinic orbit, i.e., $\Sigma = \Sigma_1 \circ \Sigma_0 : P_0 \to P_0$ where $\Sigma(x, 0, z) = (x', 0, z')$ and

$$\begin{aligned} x' &= x^* + a\mu + \alpha x z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_1\right\} + \dots \\ z' &= \quad + b\mu + \beta x z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_2\right\} + \dots \end{aligned}$$

where a, b, α, β, Φ_1 and Φ_2 are constants, and $\delta = \rho/\lambda$ is the ratio of the real parts of the eigenvalues. This map is valid only for sufficiently small z and μ, and is defined for $z > 0$ only.

There are two very different cases, $\delta > 1$ and $\delta < 1$. We found in the last lecture that the case $\delta > 1$ gives essentially the same bifurcation as the planar homoclinic case. Specifically, the bifurcation adds one periodic orbit to the dynamics, with the period of the orbit increasing monotonically (growing as log of the bifurcation parameter) as the bifurcation is approached.

The case $\delta < 1$

The return map is $\Sigma(x, 0, z) = (x', 0, z')$ with

$$\begin{aligned} x' &= x^* + a\mu + \alpha x z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_1\right\} + \ldots \\ z' &= \quad + b\mu + \beta x z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_2\right\} + \ldots \end{aligned}$$

Since $\delta < 1$ and z is small, fixed points of the map satisfy

$$\begin{aligned} x &= x^* + O(\mu, z^\delta) \\ z &= b\mu + \beta x^* z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_2\right\} + O(z, z^{2\delta}) \end{aligned}$$

(Note: Some work is required to find the order of the ignored terms in this equation for z!!)

Thus, when $\delta < 1$, we can approximate fixed points of the full 2D map by $x = x^*$ and z satisfying

$$z = b\mu + \beta x^* z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_2\right\}.$$

We look for suitable z values graphically.

We find (after some work):

- There are infinitely many fixed points if $\mu = 0$.
- These occur at $z = z_n$ where $z_n > 0$ and

  $$\cos\left\{-\frac{\omega}{\lambda}\log z_n + \Phi_2\right\} \approx 0,$$

  i.e.,

  $$-\frac{\omega}{\lambda}\log z_n + \Phi_2 \approx \frac{\pi}{2} + n\pi.$$

- It can be shown that all the fixed points are saddles at $\mu = 0$.
- There are finitely many fixed points if $\mu \neq 0$.
- As μ varies monotonically near $\mu = 0$ the fixed points appear and disappear in pairs, in saddle-node bifurcations.

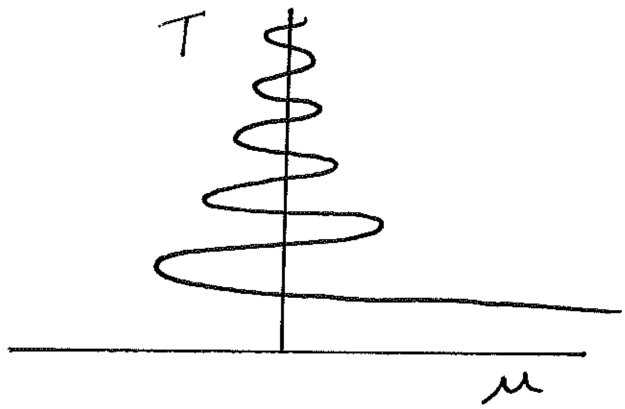
The bifurcation diagram for μ near 0 is obtained by computing the period of the periodic orbits in the flow as a function of μ. The periodic orbit in the flow corresponding to some fixed point of the map has a period that is dominated by the time spent travelling past the saddle-focus (from P_0 to P_1), i.e., (from last lecture) $T \sim -\frac{1}{\lambda} \log z$ or $z \approx \exp(-\lambda T)$ where z is the value of z for the fixed point of the map. Substituting $z \approx \exp(-\lambda T)$ into the fixed point equation

$$z = b\mu + \beta x^* z^\delta \cos\left\{-\frac{\omega}{\lambda}\log z + \Phi_2\right\}$$

gives

$$0 \approx b\mu + \beta x^* e^{-\rho T} \cos\left\{\omega T + \Phi_2\right\}.$$

Plotting this curve in the (μ, T) plane gives the bifurcation diagram for the case $\delta < 0$.



Stability of the periodic orbits can be determined using the Jacobian. It turns out that the saddle-node bifurcation at the ends of each "wiggle" in the bifurcation diagram produces a sink and saddle pair of periodic orbits (if $1/2 < \delta < 1$) or a source and saddle pair of periodic orbit (if $0 < \delta < 1/2$). The sinks/sources become saddles in period-doubling bifurcations before $\delta = 0$ is reached.

Despite all the complicated dynamics close to $\mu = 0$ in the case $\delta < 1$, the overall effect of the homoclinic bifurcation is to add one periodic orbit to the dynamics, just as in the case $\delta > 1$ or in the planar homoclinic bifurcation.

Summary A homoclinic bifurcation of a saddle-focus equilibrium for a differential equation occurs in two different ways depending on the the ratio of the eigenvalues of the linearised flow. Write $\delta = \rho/\lambda$ where the eigenvalues are $-\rho \pm i\omega$ and λ, with $\rho > 0$ and $\lambda > 0$. Then for all choices of δ, the bifurcation adds one periodic orbit to the dynamics overall. However

- if $\delta > 1$, the period of the orbit increases to infinity as the bifurcation parameter varies monotonically towards the bifurcation value;
- if $\delta < 1$, the period of the orbit increases to infinity as the bifurcation parameter wiggles about the value at which the homoclinic orbit exists.