

# THE UNIVERSITY OF AUCKLAND

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SEMESTER ONE, 2008

Campus: City

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## MATHEMATICS

Great Ideas Shaping Our World

(Time allowed: TWO hours)

### NOTE:

- This exam contains **36** questions on **12** pages. Attempt **ALL 36** questions.
- Section A contains 30 questions, each worth 2 marks, making a total of 60 marks.  
Section B contains 6 questions, each worth 10 marks, making a total of 60 marks.
- For Section A write your answers on the coloured answer sheet attached as Page 11.
- For Section B write your answers in the answer book provided.  
Answer Question **33(b)** on the coloured answer sheet attached as Page 12.  
All working must be shown.  
Start each question on a new page.
- Please do not use **red ink** in your solutions.
- Calculators are allowed for this exam.  
Stored memory on programmable calculators must be cleared before the exam starts.

## SECTION A: MULTI-CHOICE

Detach Page 11 (the coloured answer sheet) and write your name and ID number on it.  
Mark your answers with crosses in the grid provided.

**Do not write your answers to Section A in the answer book.**

1. Ten cakes are divided randomly between four people. We can be sure that everyone gets at least how many cakes?

(a) 1                      (b) 2                      (c) 0                      (d) 3

2. Which of the following numbers is a Fibonacci number?

(a) 15                      (b) 13                      (c) 14                      (d) 12

3. How many ways can the number 1011 be written as a product of primes?

(a) 2                      (b) 7                      (c) 13                      (d) 1

4. What is  $(92)^6 \pmod{9}$ ?

(a) 7                      (b) 5                      (c) 1                      (d) 3

5. Recall that a 13 digit bar-code  $d_1d_2d_3d_4d_5d_6d_7d_8d_9d_{10}d_{11}d_{12}d_{13}$  is valid if the following holds:

$$d_1 + 3d_2 + d_3 + 3d_4 + d_5 + 3d_6 + d_7 + 3d_8 + d_9 + 3d_{10} + d_{11} + 3d_{12} + d_{13} \equiv 0 \pmod{10}$$

What should the last digit of the bar-code 9 780471 47244□ be?

(a) 6                      (b) 4                      (c) 5                      (d) 3

6. Recall that a 10-digit ISBN number (used to uniquely identify published books) of the form  $d_1d_2d_3d_4d_5d_6d_7d_8d_9d_{10}$  is valid if the following holds:

$$d_1 + 2d_2 + 3d_3 + 4d_4 + 5d_5 + 6d_6 + 7d_7 + 8d_8 + 9d_9 + 10d_{10} \equiv 0 \pmod{11}$$

Which one of the following is a valid ISBN number?

(a) 0-204-00709-0      (b) 1-230-00789-0      (c) 0-204-06080-X      (d) 1-004-00700-X

7. Suppose  $X$  is a set with infinitely many elements, and  $Y$  is a subset of  $X$ . Which one of the following statements is TRUE?
- (a)  $X$  might have a greater cardinality than  $Y$
  - (b)  $X$  must have a smaller cardinality than  $Y$
  - (c)  $X$  and  $Y$  must have the same cardinality
  - (d)  $X$  must have a greater cardinality than  $Y$
8. Think about how the Pigeonhole Principle might apply to infinite sets. Suppose we have infinitely many ping-pong balls (numbered with integers starting at 0), and infinitely many barrels (numbered with integers starting at 1). All of the ping-pong balls are put in the barrels. Which one of the following must be TRUE?
- (a) One barrel must contain two ping-pong balls
  - (b) One barrel must contain infinitely many ping-pong balls
  - (c) One barrel must be empty
  - (d) None of the above must be true
9. Georg Cantor shook the foundations of mathematics (as well as philosophy and theology) with his work on infinity. Which one of the following best describes what he proved?
- (a) All infinite sets are the same size
  - (b) There are ‘more’ rational numbers than natural numbers
  - (c) There are infinitely many different sizes of infinity
  - (d) All infinite sets are different sizes
10. How many elements does the power set  $\mathcal{P}(\{1, 2, 3\})$  have?
- (a) 8                      (b) 14                      (c) 9                      (d) 3

11. Suppose we tried to make a list of all of the infinite sequences of the letters X and O.

We can use Cantor's Diagonalisation Argument to show that this is not possible; that any list will have a missing element.

Below are the first few rows and columns of an attempt to list all of the infinite sequences of the letters X and O.

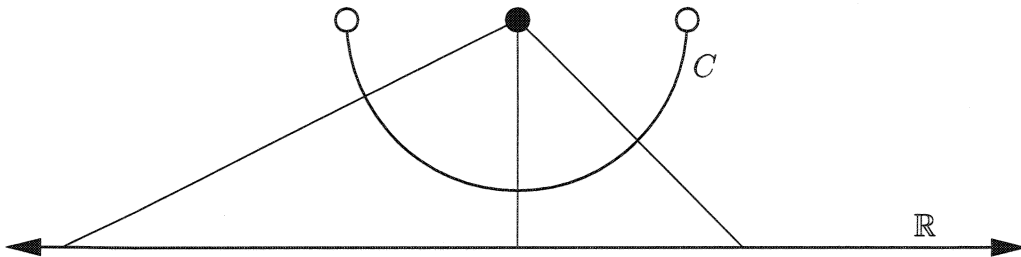
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X X X X X X X X ...
O O O O O O O O ...
X O X O X O X O ...
O X O X O X O X ...
O O X O O X O O ...
O X O O X O O X ...
X O O X O O X O ...
O X X O X X O X ...

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Which one of the following is the *beginning* of a construction of an infinite sequence we know is not in this list?

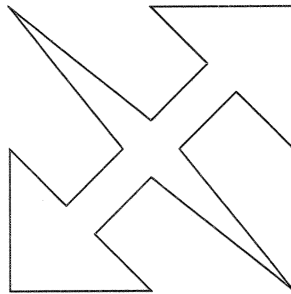
- (a) O X O O X X O O ...                      (c) X O X X O X X O ...  
 (b) X O X X O O X X ...                      (d) X O X O O O X O ...
12. The diagram below shows a curved line segment  $C$  with open ends, and the real line  $\mathbb{R}$ .



The lines radiating from the black dot give a correspondence between  $C$  and  $\mathbb{R}$ . Which one of the following is FALSE?

- (a) The curve  $C$  is the same length as the  $\mathbb{R}$   
 (b) The cardinalities of  $C$  and  $\mathbb{R}$  are the same  
 (c) Every point on  $C$  corresponds to exactly one point on  $\mathbb{R}$   
 (d) Every point in  $\mathbb{R}$  corresponds to exactly one point on  $C$

13. Consider the art gallery below.



What is the smallest number of guards which can guard this gallery?

- (a) 5                      (b) 4                      (c) 6                      (d) 3

14. Which one of the following shapes does NOT have symmetry of scale?

- (a)       (b)       (c)       (d) 

15. An icosahedron has twenty triangular faces. How many edges does it have?

- (a) 20                      (b) 30                      (c) 12                      (d) 60

16. Which of the following pairs of letters are equivalent by distortion?

- (a) A R                      (b) B Q                      (c) R B                      (d) A B

17. If the following digits are grouped so that each group is equivalent by distortion, how many groups would there be?

0    1    2    3    4    5    6    7    8    9

- (a) 3                      (b) 7                      (c) 5                      (d) 6

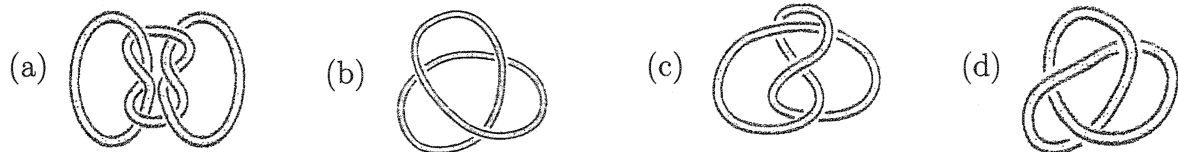
18. Which of the following letters is equivalent by distortion to K?

- (a) V                      (b) M                      (c) R                      (d) H

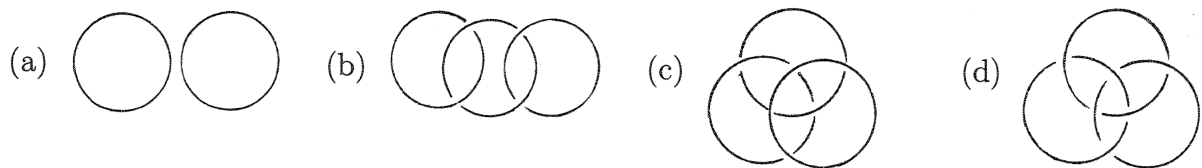
19. Which of the following best describes a loop with exactly one crossing?

- (a) Two separate loops                      (c) The figure eight knot  
(b) The unknot                      (d) A Borromean ring

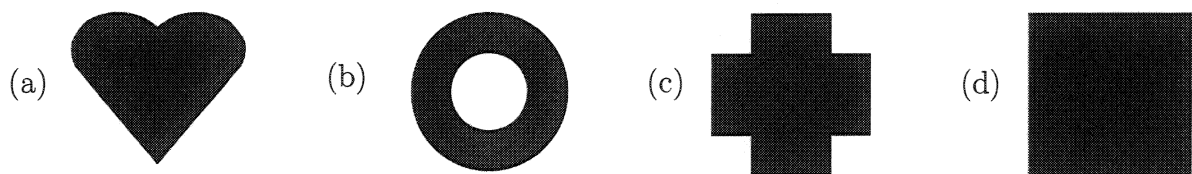
20. Which of the following knots is actually the unknot?



21. Which of the following links is a representation of the Borromean rings?



22. Brouwer's Fixed Point Theorem may NOT hold true if the discs are replaced by two sheets of what shape?



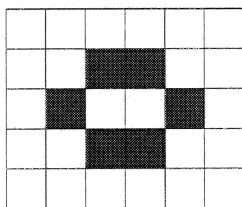
23. The Hot Loop Theorem tells us that there are two antipodal points on Earth with the same:

- (a) humidity (c) temperature and barometric pressure  
 (b) currency (d) name

24. How many faces, edges and vertices does a graphical representation of a cube have?

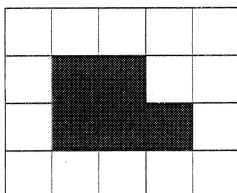
- (a) 5 faces, 12 edges and 8 vertices (c) 6 faces, 10 edges and 6 vertices  
 (b) 5 faces, 13 edges and 6 vertices (d) 6 faces, 12 edges and 8 vertices

25. What best describes the behaviour of the following population in the Game of Life?



- (a) Stable (b) Migratory (c) Explosive (d) Periodic

26. Given the following population of the Game of Life how many squares will be alive in the next generation?



- (a) 0 (b) 5 (c) 4 (d) 8

27. In the Verhulst population model  $P_{n+1} = P_n + 0.7P_n(1 - P_n)$ , what does  $P_n$  represent?

- (a) The average rate of change of population (c) The total population at year  $n$   
 (b) The population density at year  $n$  (d) The maximum sustainable population

28. Consider the equation  $f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 0.5 \\ 2 - 2x & \text{if } 0.5 < x \leq 1 \end{cases}$

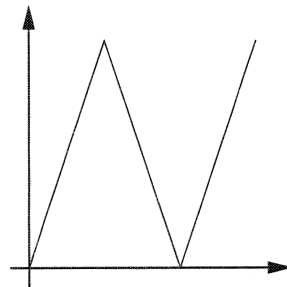
The point  $x = 0.4$  is periodic with what period?

- (a) 2                      (b) 1                      (c) 3                      (d) 4

29. Iterative systems can be sensitive to which of the following?

- (a) Initial conditions only  
(b) Accumulated errors only  
(c) Initial conditions and accumulated errors  
(d) Neither initial conditions nor accumulated errors

30. A piece of pastry is folded in 3 and stretched to its original length as shown in the graph below. How many fixed points are there?



Folding In Three

- (a) 4                      (b) 3                      (c) 2                      (d) 1

## SECTION B: LONG ANSWER

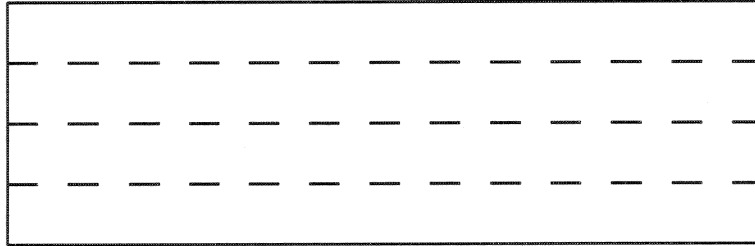
Answer each question in this section in the answer book provided.

Start your answer to each question on a new page.

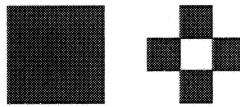
Answer Question 33(b) on the coloured answer sheet provided as Page 12.

31. (a) A student in the class EZPASS 101 uses the Pigeonhole Principle to figure out that three students in his class must share the same birthday.  
How many students must there be in EZPASS 101? (3 marks)
- (b) Express the number 200 as a sum of non-consecutive Fibonacci numbers. (4 marks)
- (c) What is the first prime number larger than 100? Why is it prime? (3 marks)
32. (a) Give a one-to-one correspondence between the natural numbers  $\mathbb{N}$  and the integers  $\mathbb{Z}$ . (5 marks)
- (b) Explain why there is NO one-to-one correspondence between:  
**EITHER:** the real numbers ( $\mathbb{R}$ ) and the natural numbers ( $\mathbb{N}$ );  
**OR:** the natural numbers ( $\mathbb{N}$ ) and their power set ( $\mathcal{P}(\mathbb{N})$ ). (5 marks)
33. (a) Draw an art gallery with 9 walls which requires exactly 3 guards, and explain why 2 guards are insufficient. (3 marks)
- (b) Answer this question on Page 12, using the Pinwheel Pattern diagram provided.
- (i) Draw a 125-unit super-tile.
- (ii) Draw the above super-tile's interior 25-unit super-tile.
- (iii) Draw the above super-tile's interior 5-unit super-tile.
- (iv) Draw the above super-tile's interior tile. (4 marks)
- (c) Draw a diagram to show that the tetrahedron is self-dual. (3 marks)

34. (a) If you took the rectangle below and joined the right and left edges with a twist, you would obtain a Möbius band. Suppose that you cut the Möbius band along the dotted lines. What would you get? (5 marks)



- (b) State the Euler characteristic of the plane and draw a simple graph to demonstrate it. (3 marks)
- (c) What is the Euler characteristic of the surface of a football? Explain why. (2 marks)
35. The first step in the collage making process of a fractal is shown below.



- (a) Draw the next step. (3 marks)
- (b) The area of the filled in square shown above is 1.
- (i) What is the area of the fractal after the first step? (2 marks)
- (ii) What is the area of the fractal after infinitely many steps? (2 marks)
- (c) What is the fractal dimension of this fractal? (3 marks)
36. A flat has three bedrooms, A, B and C and carries a rent of \$450 per week. Xavier is equally happy with each of the three rooms and thinks that a rent of \$150 per room is fair. Yvonne is equally happy with room A and C and thinks that \$200 is a fair proportion of the weekly rent for each of those rooms. However, she dislikes room B and thinks that \$50 is a fair rent for that room.
- (a) Draw two preference diagrams, one for Xavier and one for Yvonne. (4 marks)
- (b) The third flatmate Zephyr has not yet seen the flat. However, he has agreed that the rent of \$450 per week divided among the three rooms in some way is acceptable. How do we know that a rent of \$150 per week for each room will keep all three flatmates content? (6 marks)

Candidate's Name: \_\_\_\_\_ ID No: \_\_\_\_\_

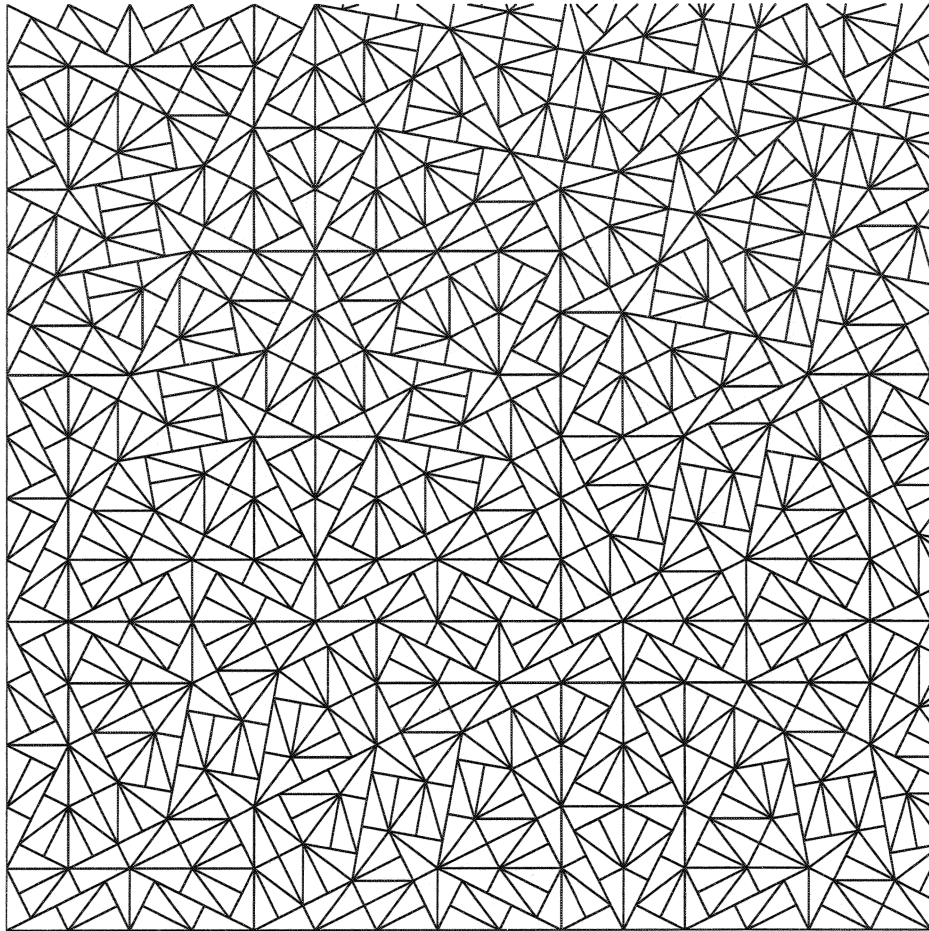
Answer Section A by putting a cross in the appropriate box for each question.  
 Marks will not be deducted for incorrect answers.

	(a)	(b)	(c)	(d)
1.				
2.				
3.				
4.				
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6.				
7.				
8.				
9.				
10.				
11.				
12.				
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14.				
15.				

	(a)	(b)	(c)	(d)
16.				
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Candidate's Name: \_\_\_\_\_ ID No: \_\_\_\_\_

Answer Question **33(b)** on the following Pinwheel Pattern diagram.



\_\_\_\_\_