Maths 190 Assignment 3 Solutions

October 9, 2010

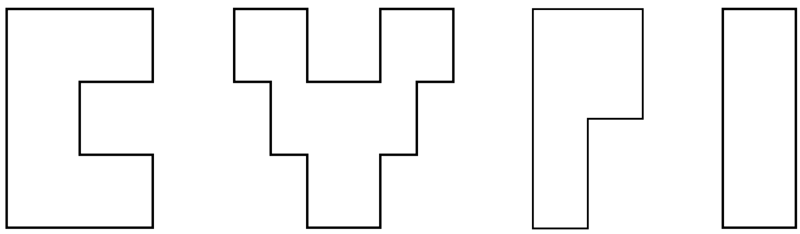
1. (4 marks) Your friend is crazy about fantasy role-playing games. He is especially fascinated about the many different dice which are so much more interesting than the ordinary ones with just six faces. One day he comes to you and asks the following questions:

   (a) "For my new adventure I want to throw a fair die to find out on which day of the week my player character will be struck by lightning. But it seems to be impossible to find dice with seven faces. Do you have an idea why no company bothers to make these?"

   Suggested answer: For a die to be fair, it must land on all faces with equal probability when it is thrown. The simplest way to make such a die would be a *regular solid*: All their faces are made of the same type of regular polygon. However, there are only five regular solids, namely the Platonic solids; neither of them has seven faces. Other ideas are hard to find: For example, it is possible to join two pyramids with a pentagonal base together to make a fair 10-faced die. But seven is not even, so this idea does not work either.

   (b) "Last night I had a dream of my player character, the mighty sorcerer Nezahualpilli of the Hag: He told me that if I count the corners and the edges of an arbitrary die he could tell me how many faces the die has. Is this some kind of black magic?"
   Suggested answer: Nezahualpilli is just a charlatan who knows the Euler formula $V - E + F = 2$. Math190 students cannot be cheated with this kind of black magic...

2. (16 marks) Claire, Paul, Vaughan and Ivo would like to give their bathrooms a slightly more personal touch. Therefore, they all decide to tile the walls only using their favourite tiles: Claire especially likes the C-shaped tiles, Vaughan only wants to see V-shaped tiles in his bathroom and Paul and Ivo also prefer tiles which slightly resemble their initials.



(a) Claire (b) Vaughan (c) Paul (d) Ivo

Figure 1: Four favourite bathroom tiles

   (a) For each of the four types of tile show one possible tiling of their bathroom walls. In each case you are only allowed to use one tile (as well as rotations or reflections of this tile). Be sure that the whole plane can be covered, in particular, leave no holes in between tiles!
   Suggested answer: Tilings using Claire's and Vaughan's tiles can be seen in Figures 2 and 3, the Ivo tiling is a bit boring, however, is very common in real tilings (Figure 4), and Paul's will be shown later, see Figure 6.

(b) For each of your four tilings, list the rigid symmetries which you can find in the tiling patterns. Indicate translations by arrows and be sure to give lines of reflection and angles of rotation.
Suggested answer:
Warning: Answers to this question depend on the tiling pattern, not noly on the tile which is used!
Each of the tiles consists of small unit squares. *Translations* are given as shifts by numbers of unit squares. Claire's pattern in Figure 2 can be shifted diagonally: Four squares to the right and four down. Vaughan's tiling patterns can both be shifted by $2\frac{1}{2}$ unit squares to the right and one down. Additionally, Vaughan's pattern on the left-hand side of Figure 3 can be shifted two unit squares down; the pattern on the right-hand side can be shifted four unit squares down. Paul's bathroom with the symmetry of scale can be shifted twelve units to the right. The
Claire's bathroom can be *rotated by 180 degrees* but has *no reflections*. Ivo's bathroom has all possible rigid symmetries, all of which are easy to see (Figure 4).

(c) Paul is proud that his tile can be used to make a pattern which has a nice symmetry of scale. Show how you can build a super-tile with 16 copies of his favourite P-shaped tile (there is more than one correct solution).
Suggested answer: Paul's super-tile is shown in Figure 5.

(d) Draw a figure which gives an example for at least one rigid symmetry in Paul's bathroom.
Suggested answer: A tiling based upon Paul's super-tile is shown in Figure 6. Shifting twelve units to the right shows that this pattern has translational symmetry.

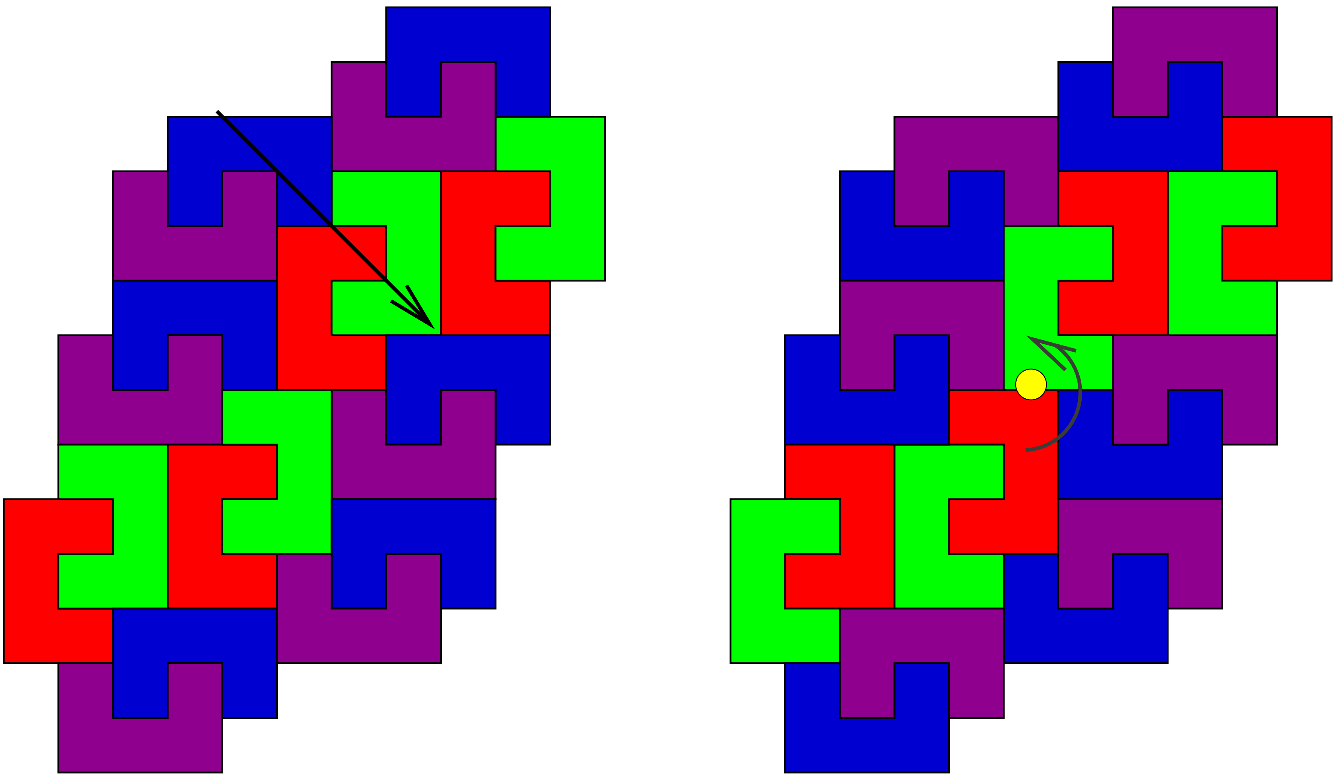


Figure 2: Claire's bathroom: A shift symmetry—four unit squares the right and four down, is shown. On the right, a part of the tiling is shown again, now rotated by 180 degrees which shows that the pattern also has rotational symmetry.

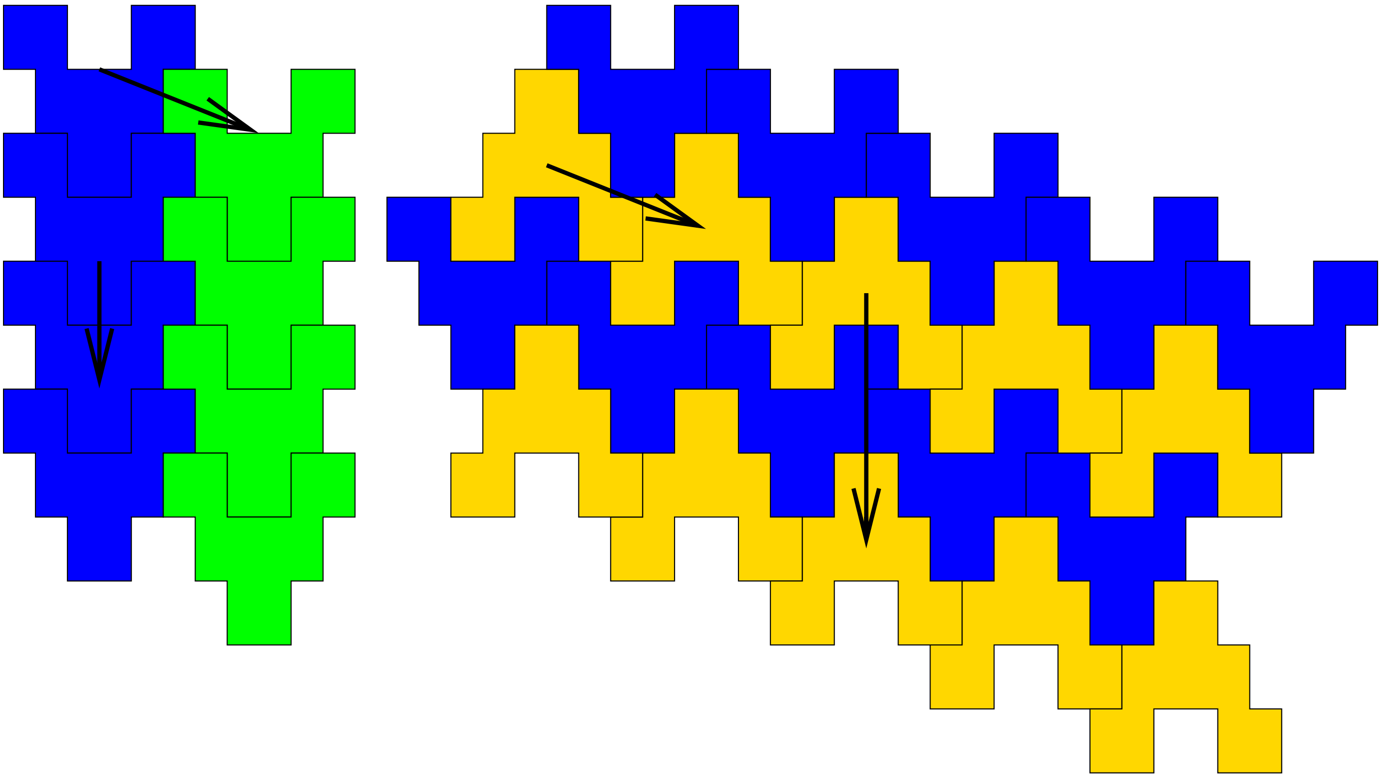


Figure 3: Two ideas for Vaughan's bathroom. Both tilings have two different translational symmetries which are indicated by arrows.

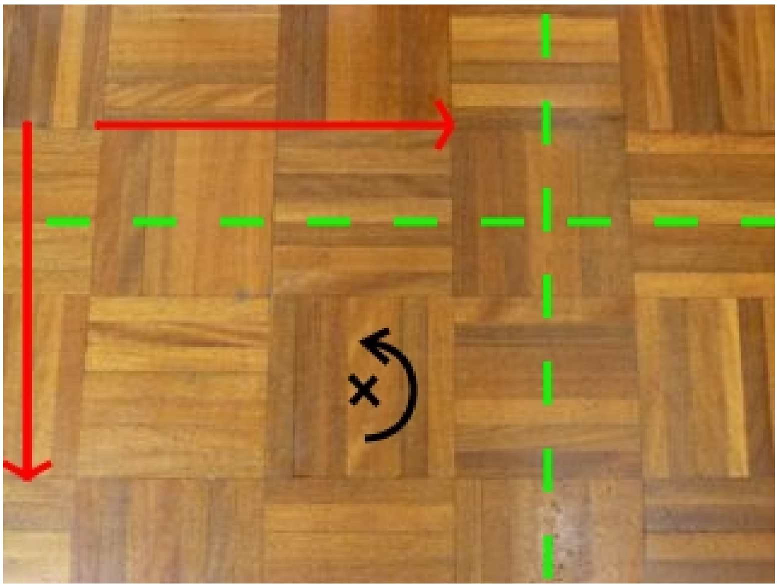


Figure 4: Ivo's tiling mostly occurs in wooden floors rather than in bathrooms. The pattern has many rigid symmetries: Red arrows indicate translational symmetry, the black arrow shows that the pattern can be rotated by 180 degrees and the two dashed green lines are the axes of reflection of a reflectional symmetry.

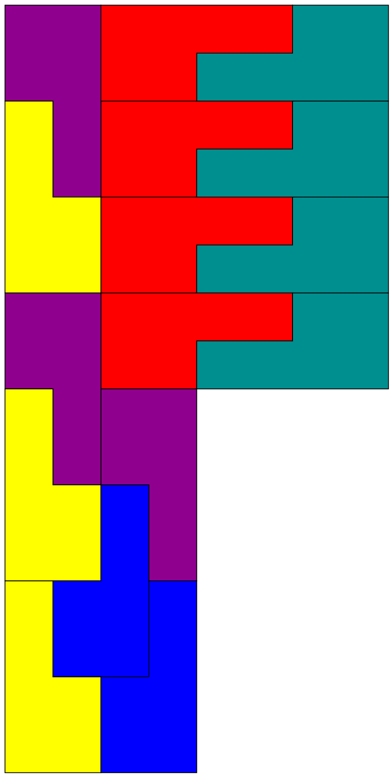


Figure 5: Paul's super-tile made from 16 copies of the P-shaped tile.

3. (8 marks) On the extra sheet you will find the first six iterations of the Hilbert curve. The Hilbert curve is a space-filling curve which consists of line segments. At each iteration, four smaller copies of the preceding iteration are assembled and glued together by additional line segments.

   (a) On the additional page, for iterations 2 to 6 indicate the four copies of the preceding iteration in a different colour. How many additional line segments are needed for glueing the copies together?
   Suggested answer: We only show colourings of the additional line segments for iterations 2 and 3, see Figure 7. One can see that always *three additional line segments* are used to join the four copies of the preceding iteration.

   (b) From the first six iterations of the Hilbert curve, what do you expect it to look like if this process is continued forever?
   Hint: This type of curve is also called a *space-filling curve.*
   Suggested answer: The term "space-filling curve" suggests that the Hilbert curve will finally fill the whole unit square. And it does...

   (c) The dimension of the unit square is 2, of course. *Without calculation*, which value do you expect for the fractal dimension of the Hilbert curve?
   Suggested answer: If the Hilbert curve really fills the unit square, its *fractal dimension* should be 2—as the "usual" dimension of the unit square. We will show in the next exercise that this is indeed true. This is some kind of a test if our idea of fractal dimension makes sense. Note that the "usual" dimension of the Hilbert curve is still 1 because it consists only of one-dimensional line segments!

4. (22 marks) Now, We shall calculate the fractal dimension of the Hilbert curve.
   Suggested answer: (a)-(d): The sentence in (a) should be ""The additional line seg-

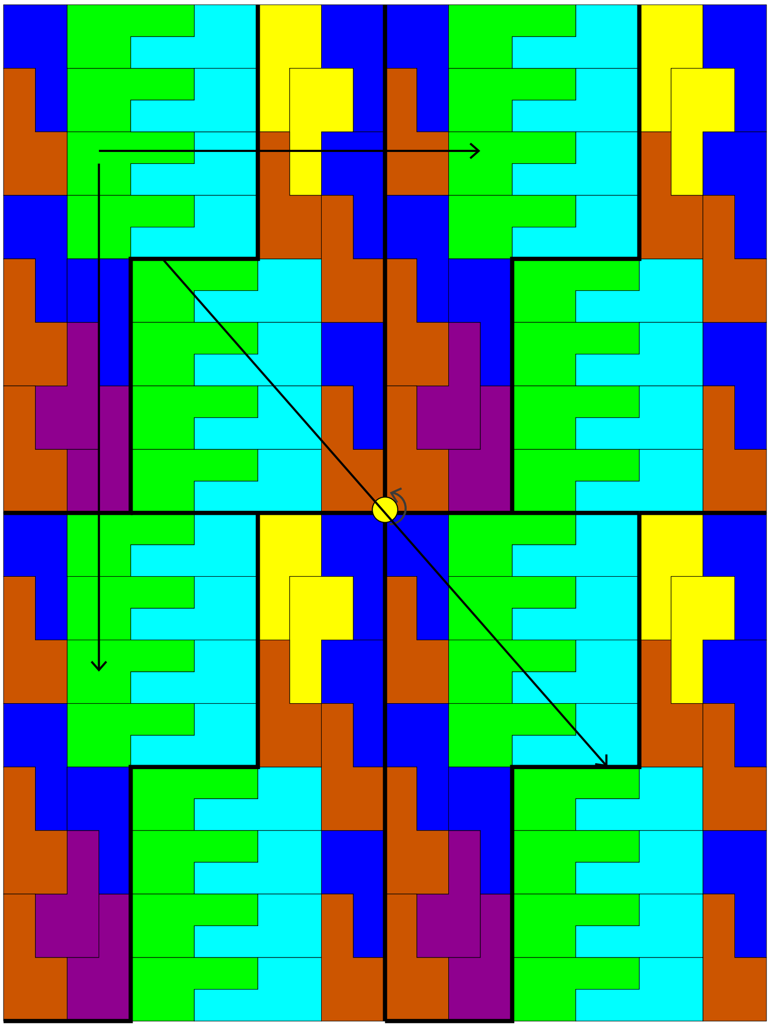


Figure 6: Paul's bathroom... with the symmetry of scale. Actually... to be correct, this is not a *full* symmetry of scale as the one we saw in the lecture. But building super-super-tiles from the super-tiles is really a bit too much.

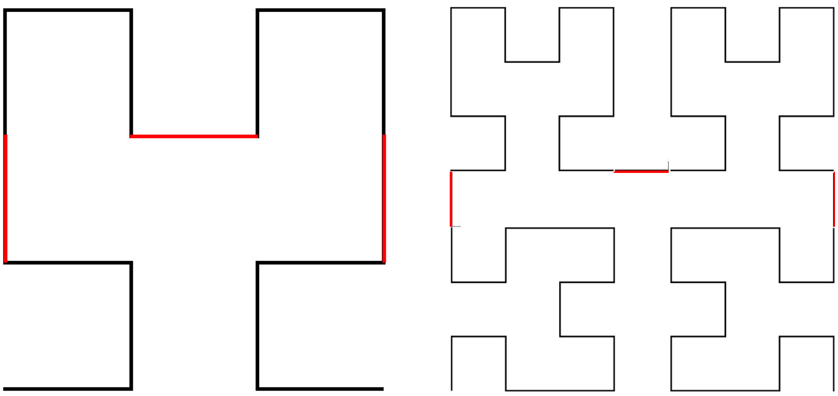


Figure 7: The additional line segments which join together copies of the preceding iterations are shown in red.

ments become less and less important for higher iterations because there are always **three** additional line segments and their length becomes **smaller** and **smaller** in comparison with the copies of the preceding iterations."
For the complete table, see Table 1. From the last column, we observe that the scaling factor tends to 2. Now, we can apply the formula for the fractal dimension from the lecture:

$$2^d = 4 \implies d = 2.$$

Considering that the Hilbert curve fills the unit square, this result is not very surprising.

Table 1: Length and scaling ratios of the Hilbert curve at subsequent iterations

Iterations	No. of line segments	Length of each line segment	total length	scaling factor
1	3	1	$3 \cdot 1 = 3$	$4 \cdot 3/5 = 2.4$
2	15	$\frac{1}{3}$	$15 \cdot \frac{1}{3} = 5$	$4 \cdot 5/9 \approx 2.22...$
3	63	$\frac{1}{7}$	$63 \cdot \frac{1}{7} = 9$	$4 \cdot 9/17 \approx 2.1176...$
4	255	$\frac{1}{15}$	$255 \cdot \frac{1}{15} = 17$	$4 \cdot 17/33 \approx 2.0606...$
5	1023	$\frac{1}{31}$	$1023 \cdot \frac{1}{31} = 33$	$4 \cdot 33/65 \approx 2.0307...$
6	4095	$\frac{1}{63}$	$4097 \cdot \frac{1}{63} = 65$	...
...	...	...	...	...
n	$4^n - 1$	$\frac{1}{2^n-1}$	$\frac{4^n-1}{2^n-1} = 2^n + 1$	$4\frac{2^n+1}{2^{n+1}+1} \approx 2$ if n large