Maths 761 Lecture 21

Topic for today
More bifurcations of fixed points in maps

Reading for this lecture
Glendinning §9.2, §9.3

Suggested exercises
Glendinning Exercises 9, #1,2,4

Today's handouts
Lecture 21 notes

More local bifurcations in maps

We again consider maps of the form

$$x_{n+1} = f(x_n)$$

where $x \in \mathbf{R}^n$ and $f : \mathbf{R}^n \to \mathbf{R}^n$ is smooth and invertible. Fixed points of a map are non-hyperbolic if one or more eigenvalues of the Jacobian lie on the unit circle.

The simplest local bifurcations have one- and two-dimensional centre manifolds. There are three main cases.

- $Df(0)$ has one eigenvalue equal to $+1$.
- $Df(0)$ has one eigenvalue equal to -1.
- $Df(0)$ has a complex conjugate pair of eigenvalues on the unit circle, but not at $+1$ or -1.

We look at these last two cases today.

A single simple eigenvalue of -1
Without loss of generality, assume that a map

$$w_{n+1} = f(w_n; \mu)$$

has a fixed point at $x = 0$ when $\mu = 0$, and that $Df(0;0)$ has a single eigenvalue of -1 with no other eigenvalues on the unit circle. Then there is a one-dimensional centre manifold on which the dynamics is of the form

$$x_{n+1} = g(x_n; \mu)$$

for $x \in \mathbf{R}$ and for some function g with

$$g(0;0) = 0 \quad \text{and} \quad \frac{\partial g}{\partial x}(0;0) = -1.$$

Expanding g in a Taylor series gives

$$g(x;\mu) = -x + g_\mu \mu + \frac{1}{2}\left(g_{xx}x^2 + 2g_{x\mu}\mu x + g_{\mu\mu}\mu^2\right) + \ldots$$

where all the derivatives are evaluated at $(0;0)$. Fixed points of the map g therefore satisfy

$$-2x + g_\mu \mu + \frac{1}{2}\left(g_{xx}x^2 + 2g_{x\mu}\mu x + g_{\mu\mu}\mu^2\right) + \ldots = 0.$$

For μ, x near zero, there is just one fixed point, at

$$x^* = \frac{1}{2}g_\mu \mu + \text{higher order terms in } \mu$$

The stability of the fixed point is determined from the Jacobian:

$$Dg(x;\mu) = -1 + g_{xx}x + g_{x\mu}\mu + \text{higher order terms}$$

At the fixed point this gives

$$Dg(x^*;\mu) = -1 + \left(\frac{1}{2}g_{xx}g_{\mu\mu} + g_{x\mu}\right)\mu + \text{higher order terms in } \mu$$

Thus the fixed point changes stability at $\mu = 0$ so long as

$$\frac{1}{2}g_{xx}g_{\mu\mu} + g_{x\mu} \neq 0.$$

It can be shown (after some more algebra) that a periodic orbit of period two bifurcates from the fixed point at $\mu = 0$ if

$$\frac{1}{2}g_{xx}^2 + \frac{1}{3}g_{xxx} \neq 0.$$

The period two orbit is stable if it coexists with an unstable fixed point and is unstable if it coexists with a stable fixed point.

In summary, a period-doubling bifurcation occurs when a fixed point of a map changes stability as one eigenvalue of the Jacobian matrix passes through the value -1. Generically, a period-two orbit will appear on one side of the bifurcation and will have the stability of the fixed point that it bifurcated from. The following theorem contains more detail.

The Period Doubling Bifurcation Theorem

Assume that the map $w_{n+1} = f(w_n; \mu)$, for $w \in \mathbf{R}^n$, has a fixed point at $w = 0$ when $\mu = 0$, and that the Jacobian matrix evaluated at $(0; 0)$ has a simple eigenvalue of -1 with no other eigenvalues on the unit circle. Then the dynamics on the centre manifold is given by $x_{n+1} = g(x_n; \mu)$, for $x \in \mathbf{R}$, with $g(0; 0) = 0$ and $\partial g/\partial x(0; 0) = -1$.

If

$$u = 2g_{\mu x} + g_\mu g_{xx} \neq 0$$

and

$$v = \frac{1}{2}g_{xx}^2 + \frac{1}{3}g_{xxx} \neq 0$$

(where the partial derivatives are evaluated at $(0, 0)$) a curve of periodic points of period two bifurcates from $(0; 0)$ into $\mu > 0$ if $uv < 0$ (or into $\mu < 0$ if $uv > 0$.)

The fixed point from which these solutions bifurcates is stable in $\mu > 0$ and unstable in $\mu < 0$ if $u > 0$, with the signs of μ reversed if $u < 0$. The bifurcating period-two orbit is stable if it coexists with an unstable fixed point and vice versa.

Example: We showed by direct computation that the following map has a period-doubling bifurcation at $\mu = 0$.

$$x_{n+1} = -(1 + \mu)x_n - x^3$$

Example: We found graphically the μ-value for which the following map (probably) has a period-doubling bifurcation.

$$x_{n+1} = \mu \sin x$$

A single simple complex conjugate pair of eigenvalues

A Hopf bifurcation occurs in a map when the Jacobian matrix evaluated at a fixed point has a simple pair of complex eigenvalues that cross the unit circle. The dynamics associated with Hopf bifurcation in maps can be much more complicated than for differential equations, depending on whether the eigenvalues cross the unit circle at roots of unity or not.

We restrict attention to the two-dimensional centre manifold. Then the general idea is that if the eigenvalues cross the unit circle at $\pm i\omega$ where $\exp(in\omega) \neq 1$ for $i = 1, 2, 3, 4$ then an invariant closed loop bifurcates from the fixed point. The fixed point changes stability at the Hopf bifurcation and the loop carries away the stability of the fixed point it bifurcates from.

This is similar to the case for Hopf bifurcation in differential equations. However, in a map the dynamics on the loop is discrete, not continuous, and there will be different behaviour depending on whether ω/π is rational or not.

See Glendinning for a statement of the Hopf Bifurcation Theorem (which this margin is too narrow to contain . . .)

Implications for bifurcations of periodic orbits in differential equations

We have looked at five types of local bifurcation for maps: saddle-node, transcritical, pitchfork, period-doubling and Hopf. Each of these corresponds to a distinct type of local bifurcation of a periodic orbit in a differential equation (or 'flow').

- A saddle-node bifurcation of periodic orbits in a flow occurs when a pair of periodic orbits of different stabilities are created as a parameter is varied.
- A transcritical bifurcation of periodic orbits in a flow occurs when a pair of periodic orbits of different stabilities pass through one another and exchange stabilities as a parameter is varied.
- A pitchfork bifurcation of a periodic orbit in a flow occurs when a periodic orbit changes stability and a new pair of periodic orbits with the same stability as each other appears.
- A period-doubling bifurcation of a periodic orbit in a flow occurs when a periodic orbit changes stability and a new periodic orbit with *double* the period of the original orbit appears. The new orbit has the stability of the orbit it bifurcated from.
- A Hopf bifurcation of a periodic orbit in a flow often produces an invariant torus. Such a torus is very delicate and soon breaks up as the bifurcation parameter is varied. This kind of bifurcation is also known as an Andronov-Hopf or secondary-Hopf bifurcation. In XPPAUT this bifurcation is given the label TR.