Maximum Size of Automorphism Groups on Compact Riemann Surfaces of Small Genus

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Abstract

Given a compact Riemann surface S with genus $g > 1$, we know that the group of automorphisms on S is isomorphic to a finite quotient Γ/Λ where Λ is the fundamental group of S and Γ is the normaliser of Λ in $\mathrm{PGL}(2, \mathbb{R})$. The groups Γ and Λ are Fuchsian groups and in particular Γ has signature $(\gamma;\, m_1, \dots, m_r)$ where γ is the orbit-genus and the m_i are the orders of the branch points. Accordingly Γ has presentation

$$\Gamma = < x_1,\, x_2, \dots x_r,\, a_1,\, b_1, \dots a_\gamma,\, b_\gamma \,|\, {x_1}^{m_1},\, {x_2}^{m_2}, \dots {x_r}^{m_r}, \prod_{i=1}^{r} x_i \prod_{j=1}^{\gamma} [a_j, b_j] >.$$

The order of the automorphism group $G = \mathrm{Aut}(S) \cong \Gamma/\Lambda$ is given by the Riemann-Hurwitz formula

$$2(g-1) = |G| \left(2(\gamma - 1) + \sum \left(1 - \frac{1}{m_i}\right)\right).$$

Conversely for every torsion-free normal subgroup Λ of finite index in Γ, there exists a compact Riemann surface S with $\mathrm{Aut}(S)$ isomorphic to Γ/Λ and with genus given by the formula above.

Let $\mu(g)$ represent the order of the largest automorphism group of a compact Riemann surface of genus g. Hurwitz showed that $\mu(g) \leq 84(g-1)$ for all g and Macbeath proved that this upper bound was sharp, in the sense that it is achieved for infinitely many values of g. Maclachlan and Accola independently proved that $8(g+1)$ is a lower bound for $\mu(g)$, and that it is sharp in the same sense.

Here we are concerned with finding the exact value of $\mu(g)$ for small g. This can be achieved by finding γ and m_i which minimize the expression in the brackets above, subject to the existence of a suitable quotient of a Fuchsian group Γ with the corresponding signature. We have done this for $2 \leq g \leq 200$ using a computer algorithm for finding normal subgroups of small index in finitely-presented groups.