Maths 761 Lecture 3 Summary

Topics for today

Invariant manifolds in linear systems
Stationary solutions in non-linear systems
Linearising nonlinear systems
Hartman–Grobman Theorem

Reading for this lecture

Glendinning: §3.3-§3.4, §4.1, §4.2, §4.5
(Harder) Glendinning: §3.1-§3.2
Strogatz: §6.1-§6.3

Recommended problems

Glendinning: §3: 8, §4: 5
Strogatz: 6.3.1-6, 6.3.11-13

Reading for next lecture

Glendinning: §4.2, §4.6

Invariant manifolds in linear systems

Consider the linear system of equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ constant matrix. The origin $\mathbf{x} = 0$ is a stationary solution for the flow. We define the following subspaces:

1. $E^u(0)$, the *unstable manifold (or subspace, or generalised eigenspace) of the origin*, is the span of the eigenvectors and generalised eigenvectors corresponding to the eigenvalues of $\mathbf{A}$ with positive real part.

2. $E^s(0)$, the *stable manifold (or subspace, or generalised eigenspace) of the origin*, is the span of the eigenvectors and generalised eigenvectors corresponding to the eigenvalues of $\mathbf{A}$ with negative real part.

3. $E^c(0)$, the *centre manifold (or subspace, or generalised eigenspace) of the origin*, is the span of the eigenvectors and generalised eigenvectors corresponding to the eigenvalues of $\mathbf{A}$ with zero real part.

The three subspaces are each invariant under the flow. The flow on each subspace is simple. For instance, all solutions on $E^u(0)$ tend to zero as $t \to -\infty$ and all solutions on $E^s(0)$ tend to zero as $t \to \infty$. (Exercise: What happens to solutions on $E^c(0)$?) The flow in the full phase space is just a superposition of the flow on the individual subspaces. These results are straightforward to prove using matrix exponentiation and linear algebra. See Glendinning §§3.1-3.3.

Linearising nonlinear systems near equilibria

Consider the nonlinear system of equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

and let $\mathbf{Df}(\mathbf{x}_0)$ be the Jacobian matrix of $\mathbf{f}$ (i.e. the matrix of first partial derivatives of $\mathbf{f}$) evaluated at an equilibrium solution $\mathbf{x} = \mathbf{x}_0$.

Under certain (rather restrictive) conditions on the eigenvalues of $\mathbf{Df}(\mathbf{x}_0)$ it is possible to find a change of coordinates $\mathbf{x} = \mathbf{h}(\mathbf{y})$ so that for $\mathbf{x}$ sufficiently close to $\mathbf{x_0}$, system (1) can be rewritten as a linear system

$$\dot{\mathbf{y}} = \mathbf{Df}(\mathbf{x}_0)\mathbf{y}.$$

See Glendinning §4.1 for details of when such a change of coordinates exists.

A more useful result (the Hartman–Grobman Theorem - see below) gives conditions under which the flow near an equilibrium in a nonlinear systems is qualitatively like that of an associated linear system, even if there is no change of coordinates transforming the nonlinear system into a linear system.

Recall that a stationary point, x_0, for a nonlinear system $\dot{x} = f(x)$ is *hyperbolic* if and only if $Df(x_0)$ has no zero or purely imaginary eigenvalues.

Theorem (Hartman–Grobman)
If $\mathbf{x} = \mathbf{0}$ is a hyperbolic stationary point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ there is a continuous invertible map (a homeomorphism), defined on some neighbourhood of $\mathbf{x} = \mathbf{0}$, which maps orbits of the nonlinear flow to those of the linear flow $\dot{\mathbf{y}} = \mathbf{Df}(\mathbf{0})\mathbf{y}$. The map can be chosen to preserve parametrisation by time.

Note: The map is continuous but not necessarily differentiable. A consequence of the lack of differentiability is that the map preserves stability but not necessarily type. It could, for instance, take orbits near a node in the nonlinear system to orbits near a spiral in the linearised system.

Aside: Big oh and little oh notation

A function $f(x)$ is $O(\alpha)$ (said "big oh of α") if

$$\lim_{x \to 0} \left| \frac{f(x)}{\alpha} \right|$$

exists. A function $f(x)$ is $o(\alpha)$ (said "little oh of α") if

$$\lim_{x \to 0} \left| \frac{f(x)}{\alpha} \right| = 0.$$

Note that any function $h(x)$ with a Taylor series expansion valid near the origin can be written

$$h(x) = h(0) + \left(\left. \frac{\partial h}{\partial x} \right|_{x=0} \right) x + O(|x|^2).$$